The Black–Litterman Model for Active Portfolio Management

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The mean-variance optimal portfolio has been criticized as counterintuitive. Often, small changes in expected returns inputted into an optimization solver can lead to big swings in portfolio positions, giving rise to extreme weightings in some assets. Jobson and Korkie [1981], Michaud [1989, 1998], Best and Grauer [1991], Chopra and Ziemba [1993], and Britten-Jones [1999], among others, argued that the hypersensitivity of optimal portfolio weights is the result of the error-maximizing nature of the mean-variance optimization. As a remedy, constraints on positions are often imposed as an alternative to prevent the optimization algorithm from driving the result towards some "extreme corner" solutions. This approach, however, is often criticized as ad hoc. Moreover, when enough constraints are imposed, one can almost pick any desired portfolio without giving too much attention to the optimization process itself. An interesting study by Jagannathan and Ma [2003], however, suggested that under some special conditions, imposing constraints is equivalent to using a Bayesian-shrunk covariance matrix or expected return forecast in the optimization process.

As an alternative remedy, Michaud [1998] proposes to interpret the efficient frontier as an uncertain statistical band rather than as a deterministic line in the mean-variance space, introducing the resampling technique as one potential way to derive a more robust resulting portfolio. Scherer [2002] points out, however, some of the pitfalls of the resampling methodology, such as the possibility of the resampled frontier moving from concave to convex. Harvey et al. [2006] also showed that the resampling methodology implicitly assumes that the investor has abandoned the maximum expected utility framework. In addition, the resulting resampled efficient frontier is shown to be suboptimal as dictated by Jensen's inequality. In Harvey, Liechty, and Liechty [2008], the Bayesian approach to portfolio selection was shown to be superior to the resampling approach.

Since the Black–Litterman (BL) model appeared in the literature as Black and Litterman [1991, 1992], it has received considerable interest from the investment management industry. Unlike the resampling technique, which introduces noise into the efficient frontier, the BL framework takes an entirely different route based on Bayesian analysis in solving the error maximization problem. The BL framework points out that, because assets are correlated, changes in some assets' expected excess returns due to active investment views should also lead to revisions of expected excess returns of assets that are not explicitly involved in the active investment views. Take a global portfolio of stock and bond markets as an example. If expected excess returns of the U.S. stock market are revised upward, then the expected excess returns of all assets and portfolios of assets that are correlated with the U.S. stock market should also be revised in a direction that is consistent with the covariance matrix of the assets. As such, the error in estimating expected excess return in one asset, if any, will be extended to all other correlated assets, so that a robust optimal portfolio can be derived when these revised inputs are fed into the optimization process.

Many studies inspired by this framework further advance our understanding and implementation of the Black–Litterman framework. Lee [2000] and Satchell and Scowcroft [2000] further elaborated and expanded the theoretical framework, while others, such as Bevan and Winkelmann [1998], He and Litterman [1999], Herold [2003], Idzorek [2004], and Jones, Lim, and Zangari [2007] focused on implementation.

Many practitioners seem to suggest that one of the key contributions of the BL framework is the derivation of implied equilibrium excess returns from a given portfolio through reverse optimization. For instance, in the investment industry, many clients start with their strategic predetermined benchmark portfolio. Given the benchmark portfolio weights and a scaling parameter, one can easily derive the benchmark implied expected excess returns of the assets through reverse optimization, which may be interpreted as the equilibrium views employed as the starting point for subsequent active investment analysis. To the best of our knowledge, however, Sharpe [1974] is the first to have provided insights on this subject matter. In our opinion, the key element of the BL framework is the combination of active investment views and equilibrium views through a Bayesian approach, which has been shown to result in more robust portfolios that are less sensitive to errors in expected excess return inputs. As active views are involved, by definition, the framework has to be analyzed and understood within the context of active management—namely, beating the benchmark within a certain tracking error.

This article adds to the literature by first pointing out that the Black–Litterman framework was derived under the mean-variance portfolio efficiency paradigm, which is different from the common objective in active management, namely, maximizing the active alpha for the same level of active risk. We show, by presenting and analyzing resulting portfolio statistics, how the inconsistencies lead to unintentional trades and risks when the framework is implemented at face value. Finally, we consider potential remedies.

REVIEW OF BLACK–LITTERMAN FRAMEWORK

Suppose there are N assets and K active investment views. The original Black–Litterman model of expected excess returns in Black and Litterman [1991, 1992] was expressed as

$$\mu = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\Pi + P'\Omega^{-1}Q]$$

where μ is an $N\times$ vector of expected excess returns, τ is a scaling parameter, Σ is an $N\times N$ covariance matrix, P is a $K\times N$ matrix whose elements in each row represent the weight of each asset in each of the K-view portfolios, Ω is the matrix that represents the confidence in each view, and Q is a $K\times 1$ vector of expected returns of the K-view portfolios. A view portfolio may include one or more assets through nonzero elements in the corresponding elements in the P matrix. Several papers and articles discuss in detail how to formulate active investment views in the Black–Litterman framework; for examples, see Lee [2000], Idzorek [2004], and Jones, Lim, and Zangari [2007].

By applying the Matrix Inversion Lemma, the original Black–Litterman equation can be rewritten in a more intuitive way, as follows:

$$\mu = \Pi + \Sigma P'\left[\frac{\Omega}{\tau} + P\Sigma P'\right]^{-1}(Q - P\Pi) = \Pi + V \quad (1)$$

where V is a term that captures all deviation of expected excess returns from the equilibrium due to active investment views,

$$V = \Sigma P'\left[\frac{\Omega}{\tau} + P\Sigma P'\right]^{-1}(Q - P\Pi) \quad (2)$$

Equation (1) helps expose the intuition behind the Black–Litterman framework. Under the BL framework, the expected excess return of assets is equal to the assets' equilibrium excess return, Π, plus a term that captures the deviation of our views of the K portfolio of assets, Q, from the equilibrium implied views, $P\Pi$. Therefore, the expected excess return will be different from the equilibrium excess return if, and only if, our investment views are not redundant to, or implied by, the equilibrium views.

OPTIMAL ACTIVE MANAGEMENT

After expected excess returns are derived, a risk target needs to be defined in order to determine the final active weights. In active management, active return, or what is commonly known as alpha, is typically defined as the return of the active portfolio in excess of the benchmark portfolio. Active risk is defined as the standard deviation of alpha, also known as tracking error. In a nutshell, the objective of active management is to maximize alpha for a given level of tracking error. In other words, active management attempts to maximize the information ratio, IR, defined as the ratio of alpha to tracking error. For example, this objective is reflected in Bevan and Winkelmann [1998]:

> After finding expected returns, we then set target risk levels. Since we construct our optimal portfolio relative to a benchmark, we consider all of our risk measures as risks relative to the benchmark. The two risks that we care most about are the tracking error and the Market Exposure (p. 5).

In this section, we provide an analytical framework for determining the optimal active positions given the objective of maximizing the information ratio.

Definitions:

ω_B	vector of benchmark portfolio weights
ω_a	vector of active positions
ω	vector of active portfolio weights, which is the sum of ω_B and ω_a
ω_{GMV}	vector of weights of the global minimum variance portfolio, GMV
μ_{GMV}	expected excess return of the global minimum variance portfolio, GMV
γ	scaling parameter
λ	active risk aversion parameter
θ	Lagrangian multiplier

Recall that the objective function of active management in the presence of a benchmark is to maximize the total return of the portfolio with a penalty on the square of tracking error; that is,

$$\max\ (\varpi_a + \varpi_B)'\mu - \lambda\varpi_a'\,\Sigma\varpi_a \tag{3}$$

$$\text{s.t. } \varpi_a'\,1 = 0$$

It is easy to show that the previous solution also maximizes the information ratio. Taking the first derivative of the Lagrangian gives

$$\mu - 2\lambda\Sigma\varpi_a - \theta 1 = 0$$

$$\varpi_a = \frac{1}{2\lambda}\Sigma^{-1}(\mu - \theta 1) \tag{4}$$

Substituting Equation (4) into the budget constraint in the objective function gives

$$\frac{1}{2\lambda}(\mu'\Sigma^{-1}1 - \theta 1'\Sigma^{-1}1) = 0$$

that is,

$$\theta = \frac{1'\Sigma^{-1}}{1'\Sigma^{-1}1}\mu = \varpi_{\text{GMV}}'\mu = \mu_{\text{GMV}}$$

Combining with Equation (4) gives the optimal vector of active positions as

$$\varpi_a = \frac{1}{2\lambda}\Sigma^{-1}(\mu - \mu_{\text{GMV}}1) \tag{5}$$

Alternatively, Equation (5) can be expressed as follows:

$$\varpi_a = \frac{1}{2\lambda}\Sigma^{-1}(I - 1\varpi_{\text{GMV}}')\mu \tag{6}$$

Equation (5) offers intuitive economic meanings. In optimizing the IR, the process makes multiple pairwise comparisons of the return of each asset against the return of the global minimum variance portfolio, GMV. Long positions are taken for assets that are expected to outperform the GMV portfolio, and vice versa. The vector of optimal active weights is the result of the risk-adjusted combination of all of these pair trades.

EXHIBIT 1

Summary Statistics of Two-Asset Example: Stocks and Bonds

	Market-Cap Weight	Volatility	Correlation	Equilibrium Excess Return	Black–Litterman Excess Return
Stocks	75%	13%	0.3	6.46%	2.39%
Bonds	25%	5%		1.02%	1.17%

Example

To put the discussion in context, consider the following oversimplified example in applying the Black–Litterman framework.

Suppose there are only two asset classes in the benchmark portfolio—stocks and bonds—with benchmark weights, volatilities, and correlation as reported in Exhibit 1. To derive the equilibrium views, the literature, including Bevan and Winkelmann [1998], He and Litterman [1999], Drobetz [2001], Idzorek [2004], and Jones, Lim, and Zangari [2007], assumes that the benchmark portfolio is a mean-variance efficient portfolio. As a result, implied equilibrium excess returns can be derived by a reverse optimization from the benchmark weights according to

$$\Pi = \gamma \Sigma \varpi_B \tag{7}$$

where, in our example, Π is the 2×1 vector of equilibrium excess returns, γ is a risk aversion parameter, Σ is the 2×2 covariance matrix, and ω_B is the 2×1 vector of market-capitalization benchmark weights.[1] We set the value of γ such that the resulting equilibrium excess returns will provide an expected Sharpe ratio of 0.5 for the portfolio. Given these parameters, the equilibrium excess returns for stocks and bonds are found to be 6.46% and 1.02%, respectively.

Next, we assume that there is only one active investment view—stocks are expected to underperform bonds by 3%. In matrix notation, this view can be expressed as

$$P\mu = Q \tag{8}$$

where $P = [1 \ -1]$ and $Q = -3\%$. To set the confidence of the view, we followed the suggestion of He and Litterman [1999] by using

$$\frac{\Omega}{\tau} = \mathrm{diag}(\mathrm{diag}(P\Sigma P')) \tag{9}$$

We then applied Equation (1) to derive the Black–Litterman expected excess returns of stocks and bonds at 2.39% and 1.17%, respectively. All results are summarized in Exhibit 1.

Note that because the active view is bearish on stocks relative to bonds, the final expected premium of stocks over bonds becomes 1.22% (2.39% – 1.17%) versus the equilibrium premium of 5.44% (6.46% – 1.02%).

Lastly, for the sake of illustration, we set the value of λ in Equation (6) so that the resulting active positions give a tracking error of 2%. The optimal active weights are determined to be +16% stocks and –16% bonds, respectively.

The results are interesting, if not surprising. The only active view in this example is a bearish view on stocks versus bonds. Why would the active positions overweight stocks and underweight bonds? We explore this question more fully next.

PROBLEMS OF APPLYING BLACK–LITTERMAN IN ACTIVE MANAGEMENT

The original Black–Litterman model was derived under the mean-variance equilibrium framework, which attempts to maximize return for a certain level of portfolio risk measured by standard deviation, or volatility. The objective of this section is to illustrate that strict application of the Black–Litterman framework in active management can potentially lead to unintentional trades.

As we previously discussed, the objective of active management is to maximize the information ratio (IR). Consider the case where we do not have any investment views, such that the vector of expected excess return is just the equilibrium. Presumably, the only action that makes sense in this informationless case is to just hold the benchmark portfolio and make no active trades. However, the following analysis will show that, surprisingly, IR maximization will lead to active trades in this example.

As suggested by the literature and previously explained in this article, it has become standard procedure to derive the benchmark-implied equilibrium excess return, or Π, through reverse optimization, according to Equation (7), as $\gamma\Sigma\omega_B$.

The optimal vector of active positions, given equilibrium assumptions and no active views in this case, can be derived by simply substituting $\gamma\Sigma\omega_B$ into the expected excess return in Equation (6); that is,

$$\begin{aligned}\varpi_{a,\Pi} &= \frac{1}{2\lambda}\Sigma^{-1}(\gamma\Sigma\varpi_B - \varpi'_{\mathrm{GMV}}\gamma\Sigma\varpi_B 1) \\ &= \frac{\gamma}{2\lambda}(\varpi_B - \Sigma^{-1}\varpi'_{\mathrm{GMV}}\Sigma\varpi_B 1) \qquad (10)\end{aligned}$$

Notice that $\varpi'_{\mathrm{GMV}}\Sigma\varpi_B$ is a scalar equal to the covariance of the global minimum variance portfolio, GMV, and the benchmark portfolio, B. Together with the investment budget condition of

$$1'\varpi_B = \frac{1}{\gamma}1'\Sigma^{-1}\Pi = 1$$

we can determine that

$$\varpi'_{\mathrm{GMV}}\Sigma\varpi_B = \frac{1'\Sigma^{-1}}{1'\Sigma^{-1}1}\Sigma\varpi_B = \frac{1}{1'\Sigma^{-1}1} = \sigma^2_{\mathrm{GMV}} \qquad (11)$$

In fact, it can be shown that the covariance of any given portfolio with the global minimum variance portfolio GMV is equal to the variance of GMV (see Grinold and Kahn [1999] for details).

Substituting Equation (11) into Equation (10), the optimal vector of active positions under the no-alpha information scenario can be given as

$$\varpi_{a,\Pi} = \frac{\gamma}{2\lambda}\left(\varpi_B - \frac{\Sigma^{-1}1}{1'\Sigma^{-1}1}\right) = \frac{\gamma}{2\lambda}(\varpi_B - \varpi_{\mathrm{GMV}}) \qquad (12)$$

Equation (12) suggests that unless the client chooses the GMV as the benchmark portfolio, such that $\varpi_B = \varpi_{\mathrm{GMV}}$, use of the Black–Litterman model will generate a set of active trades, even in a case such as this of no investment information. In particular, the vector of active trades is a positive scalar multiple of the difference between the benchmark portfolio and the GMV.

This apparently counterintuitive result is related to the mismatch in objective function between mean-variance portfolio efficiency, which attempts to maximize the Sharpe ratio (SR), versus the alpha tracking-error efficiency, which attempts to maximize the information ratio instead. Some discussion on this topic appears in Roll [1992] and Lee [2000, Ch. 2]. Recall that the implied equilibrium excess return in Equation (7) is the result of reverse-optimizing the benchmark portfolio weights under the maximum-SR criteria. In general, all else equal, the higher the volatility of an asset, the higher will be the implied equilibrium excess return.

The step in approaching the maximum-IR objective is where inconsistency emerges. Notice that under the maximum-IR criteria, no attention is paid to overall portfolio volatility. Instead, any discrepancies in pairs of asset returns are perceived as *alpha opportunities* and, therefore, a portion of the total tracking error budget will be allocated to these opportunities. As a result, what seems to be at equilibrium under the max-SR criteria is, by definition, at disequilibrium, and active trades are then initiated to restore the portfolio to the max-IR condition. In the previous example in which stocks and bonds are the only portfolio assets, the implied equilibrium return of stocks is higher than for bonds under the max-SR criteria. Therefore, even in the absence of an active view, under the IR criteria, the long-stocks/short-bonds active trade is perceived as an alpha-generating trade, even though this trade embeds absolutely no alpha information at all.

In the following sections, we further elaborate on how the IR criteria perceive discrepancies in equilibrium returns as alpha opportunities and, furthermore, lead to a portfolio that is more risky than the benchmark portfolio.

INFORMATIONLESS VIRTUAL ALPHA

In this section, we derive some performance statistics for the active portfolio within this informationless environment. Given no investment views, all expected excess returns are equal to the implied equilibrium expected excess returns from the benchmark portfolio.

Alpha

Perceived alpha in this example can be calculated as follows:

$$\begin{aligned}\alpha &= \varpi_a' \Pi \\ &= \frac{\gamma}{2\lambda}(\varpi_B - \varpi_{\mathrm{GMV}})'\gamma\Sigma\varpi_B \\ &= \frac{\gamma^2}{2\lambda}(\varpi_B' \Sigma\varpi_B - \varpi_{\mathrm{GMV}}'\Sigma\varpi_B) \\ &= \frac{\gamma^2}{2\lambda}(\sigma_B^2 - \sigma_{B,\mathrm{GMV}}) \\ &= \frac{\gamma^2}{2\lambda}(\sigma_B^2 - \sigma_{\mathrm{GMV}}^2)\end{aligned} \tag{13}$$

Tracking Error

Active risk, measured by tracking error, in this informationless example, is given by

$$\begin{aligned}TE &= \sqrt{\varpi_a' \Sigma\varpi_a} \\ &= \frac{\gamma}{2\lambda}\sqrt{(\varpi_B - \varpi_{\mathrm{GMV}})'\Sigma(\varpi_B - \varpi_{\mathrm{GMV}})} \\ &= \frac{\gamma}{2\lambda}\sqrt{\sigma_B^2 + \sigma_{\mathrm{GMV}}^2 - 2\sigma_{B,\mathrm{GMV}}} \\ &= \frac{\gamma}{2\lambda}\sqrt{\sigma_B^2 - \sigma_{\mathrm{GMV}}^2}\end{aligned} \tag{14}$$

Information Ratio

Therefore, the information ratio is given by

$$IR = \frac{\alpha}{TE} = \gamma\sqrt{\sigma_B^2 - \sigma_{\mathrm{GMV}}^2} \tag{15}$$

Equation (15) reveals the interesting result that the perceived information ratio is proportional to the square root of the difference in the variance of the benchmark portfolio and the global minimum variance portfolio. Consequently, the riskier the benchmark portfolio, the larger will be the virtual alpha opportunity that is perceived by applying the BL framework, and the more resulting active trades are executed.

Beta to Benchmark

We first determine the covariance between the active weights and the benchmark as follows:

$$\begin{aligned}\sigma_{a,B} &= \varpi_a' \Sigma\varpi_B \\ &= \frac{\gamma}{2\lambda}(\varpi_B - \varpi_{\mathrm{GMV}})'\Sigma\varpi_B \\ &= \frac{\gamma}{2\lambda}(\sigma_B^2 - \sigma_{\mathrm{GMV}}^2)\end{aligned} \tag{16}$$

The active beta with respect to the benchmark is then given by

$$\begin{aligned}\beta_a &= \frac{\sigma_{a,B}}{\sigma_B^2} \\ &= \frac{\gamma}{2\lambda}(\varpi_B - \varpi_{\mathrm{GMV}})'\Sigma\varpi_B \\ &= \frac{\gamma}{2\lambda}\left(1 - \frac{\sigma_{\mathrm{GMV}}^2}{\sigma_B^2}\right) \\ &> 0\end{aligned} \tag{17}$$

That is, the informationless active positions lead to an unintentional net exposure to the benchmark. In other words, the set of active trades together implies a bullish view on the benchmark portfolio so that alpha tends to be positive when the benchmark portfolio delivers a positive return.

The portfolio beta can be determined similarly,

$$\begin{aligned}\beta &= \frac{(\varpi_a + \varpi_B)'\Sigma\varpi_B}{\sigma_B^2} \\ &= 1 + \beta_a \\ &> 1\end{aligned} \tag{18}$$

Volatility

The variance of the active portfolio can also be derived as follows:

$$\sigma^2 = (\varpi_a + \varpi_B)' \Sigma(\varpi_a + \varpi_B)$$
$$= TE^2 + \sigma_B^2 + 2\sigma_{a,B}$$

Substituting Equations (14), (16), and (17) gives

$$\begin{aligned}\sigma &= \sqrt{\frac{\gamma}{2\lambda}(\sigma_B^2 - \sigma_{\mathrm{GMV}}^2)\left(\frac{\gamma}{2\lambda}+2\right)+\sigma_B^2} \\ &= \sigma_B\sqrt{\beta_a\left(\frac{\gamma}{2\lambda}+2\right)+1} \\ &> \sigma_B\end{aligned} \tag{19}$$

In summary, in this informationless example, the mismatch of the objective function between max-SR and max-IR leads the active manager to believe that a positive information ratio is available and, thus, the active manager initiates a set of active trades, which leads to an active portfolio that is more volatile than the benchmark portfolio and that has a positive net-beta exposure.

BLACK–LITTERMAN MODEL AND ACTIVE MANAGEMENT

Implementing the Black–Litterman model in active management simply requires substituting the expected excess returns from the Black–Litterman framework in Equation (1) into the optimal active positions in Equation (6), so that

$$\varpi_a = \frac{1}{2\lambda}\Sigma^{-1}\left[(\Pi + V) - \varpi'_{\mathrm{GMV}}(\Pi + V)1\right]$$

which can be grouped into two terms, as follows:

$$\begin{aligned}\varpi_a &= \frac{1}{2\lambda}\Sigma^{-1}(\Pi - \varpi'_{\mathrm{GMV}}\Pi 1) + \frac{1}{2\lambda}\Sigma^{-1}(V - \varpi'_{\mathrm{GMV}}V1) \\ &= \varpi_{a,\Pi} + \varpi_{a,V}\end{aligned} \tag{20}$$

The first term in Equation (20) corresponds to the case of using the equilibrium expected excess returns as inputs to achieve the max-IR objective. Therefore, it is equivalent to the case of the informationless scenario discussed earlier in which the max-IR objective will still lead to active trades as given by Equation (12).

The second term in Equation (20) has a similar functional form, except that the expected excess returns in the parentheses, V, are determined by the deviation of any investment views on portfolios P from the views which are implied by the equilibrium, $P\Pi$. The details are given in Equation (2).

Consider the case of no investment view, such that Q and $P\Pi$ are the same, or, in other words, such that V is zero. In this case, the optimal vector of active trades is the same as the case in which the equilibrium excess returns are used, as depicted in Equation (12).

In the presence of investment views, such that Q is different from $P\Pi$, the relative importance of the two terms in Equation (20) largely depends on confidence in the investment views. For instance, for investment views that reflect very low confidence, such that $\Omega \to \infty$, the second term in Equation (20) approaches zero. The resulting optimal active positions once again converge to the case of no investment view.

The more interesting and relevant case is that in which the investment views, although uncertain, come with some meaningful degrees of confidence. The exact values of the two components in Equation (20) depend on, among other moving parts, how one specifies the confidence of views relative to equilibrium. For example, Equation (9), suggested by He and Litterman [1999], gives rise to the following expression for V:

$$V = \frac{1}{2}\Sigma P'(P\Sigma P')^{-1}(Q - P\Pi) \tag{21}$$

In general, the equilibrium component, Π, is not a negligible component of the vector of expected excess returns. As a result, the unintentional, informationless component of active trades, $\omega_{a,\Pi}$, plays a nontrivial role in active management when the Black–Litterman model is applied.

By now, it should be clear why, in the example using stocks and bonds, the optimal tactical trade is to go long stocks and short bonds even when the only investment view is relatively bearish on stocks. When the confidence assigned to the active investment view is not particularly strong, the equilibrium relative returns can become so dominating that they drive most of the tactical positions in the portfolio, even when they do not represent any relevant investment information.

Remedies

As the previous discussion highlighted, the root cause of an inconsistency in applying the Black–Litterman framework to active portfolio management is the mismatch between the optimization problem used in backing out the equilibrium implied excess returns (i.e., an unconstrained SR optimization) and the optimization problem used to construct an active portfolio (i.e., a constrained IR optimization). The most obvious way to fix this problem is to make the two optimization problems consistent. Active management cannot be achieved in an SR optimization framework because the manager's active bets will not be independent of the benchmark portfolio when some constraints are binding (see Roll [1992] for an example). Therefore, a practical remedy is to back out the equilibrium implied excess returns using the same IR optimization problem employed in constructing an active portfolio. This is done by replacing Π that solves the reverse SR optimization problem, as in Equation (7), with Π that implicitly solves the IR optimization problem intended to be used in building the active portfolio (i.e., reverse-optimizing the problem expressed in Equation (3), with Π replacing μ and imposing any additional constraints applicable to the portfolio). Formally, Π implicitly solves

$$0 = \arg\max_{\varpi_a} (\varpi_a + \varpi_B)'\Pi - \lambda\varpi_a' \Sigma\varpi_a \qquad (22)$$
$$\text{s.t. all other constraints}$$

instead of Π, as suggested in Black–Litterman [1992], that solves the following reverse SR optimization problem

$$0 = \arg\max_{\varpi_a} (\varpi_a + \varpi_B)'\Pi - \lambda(\varpi_a + \varpi_B)'\Sigma(\varpi_a + \varpi_B)$$
$$\text{s.t. } \textit{no} \text{ constraint}$$

Solving Equation (22) explicitly at first seems difficult in the presence of other constraints in the optimization. It turns out, however, that if we choose any Π whose elements are all the same, such as

$$\Pi_i = \Pi_j \quad \forall i \neq j$$

then the first term in the objection function of Equation (22) drops out of the optimization as it becomes constant (recall that we have constraint $\varpi_a'1 = 0$), and we are left with a tracking error minimization problem. Clearly, we can minimize tracking error by setting $\varpi_a = 0$ (i.e., taking no active weight). In other words, any constant vector of expected excess return Π always implicitly solves Equation (22) regardless of any other constraints in the problem. This observation is very intuitive. To ensure that no unintentional bet is made in an active portfolio in the absence of any active view, the prior belief for expected excess returns of the assets should be an uninformative one—that is, all assets are expected to have the same excess returns.

Of all the possible values of prior expected return, the most intuitive one is to set Π equal to 0 where all assets are expected to yield a risk-free rate of return as priors. Herold [2003] is the only other study to our knowledge that uses zeros as the equilibrium prior for active portfolio management. This choice of Π leads naturally to the portable alpha implementation for active portfolio management. Because we can drop the benchmark weights from the IR optimization problem and focus exclusively on optimizing active portfolio weights,

$$\max_{\varpi_a} \varpi_a'\mu - \lambda\varpi_a' \Sigma\varpi_a \qquad (23)$$
$$\text{s.t. all other constraints}$$

where $\mu = V$ from setting $\Pi = 0$ in Equation (1), we can build any active portfolio by focusing only on constructing an alpha overlay portfolio. The total portfolio will simply be the sum of the benchmark and the alpha overlay portfolio, $\varpi = \varpi_a + \varpi_B$.

For example, suppose we are managing a long-only portfolio that allows leverage through borrowing up to 5% of asset value and with the S&P 500 as the portfolio's benchmark. Suppose further that we form active investment views of securities in the portfolio and express them in the Black–Litterman framework in the view matrix P, view expected excess returns vector Q, and view confidence matrix Ω. The portable alpha implementation using the Black–Litterman framework can be achieved by first calculating expected excess returns of the assets μ using Equation (1) with Π set to zero, and then using these returns as inputs to the IR optimization problem in Equation (23) to solve for active weights ω_a with the constraints,

$$0 \leq \varpi_a' 1 \leq 5\% \quad (< 5\% \text{ leverage})$$
$$\varpi_a \geq -\varpi_B \quad (\text{long-only constraint})$$

EXHIBIT 2
Revised Summary Statistics of Two-Asset Example: Stocks and Bonds

	Market-Cap Weight (Benchmark)	Volatility	Correlation	Equilibrium Excess Return (Π)	Black–Litterman Active Excess Return
Stocks	75%	13%	0.3	**0.0%**	**–1.45%**
Bonds	25%	5%		**0.0%**	**+0.05%**

Adding the benchmark weight ω_B to the active weight ω_a will result in a final portfolio that respects all constraints.

To further illustrate our point, we can go back to our original example and apply the remedies just proposed. In other words, we now set $\Pi = 0$, keeping all other estimates the same, and apply Equation (1) to derive the expected active returns resulting from our bearish investment view on stocks versus bonds. Exhibit 2 reports the revised values.

We can now calculate the reverse-optimized active portfolio associated with the newly calculated active excess returns and scale it so that the final portfolio meets the defined target tracking error. It is easy to verify that the final active portfolio will be composed of a short position in stocks (–16.1%) and a long position in bonds (+16.1%) for a total tracking error of 2%. Consequently, the final total portfolio weights are +58.9% (75% – 16.1%) and +41.1% (25% + 16.1%) on stocks and bonds, respectively. As expected, the resulting portfolio underweights stocks and overweights bonds relative to the benchmark and is thus more intuitive given our initial investment view.

CONCLUSION

Of the various potential remedies to the hypersensitivity of a mean-variance optimal portfolio with respect to changes in inputs, the Black–Litterman framework stands out as the most theoretically sound and elegant of all. In the early days after this framework was introduced, it was often interpreted as an asset allocation model, or as an expected-return forecasting model. In our view, the BL framework is a portfolio construction process based on an elegant application of Bayesian analysis in combining different sources of input estimates. While we are fascinated by the strong theoretical underpinning of this framework, rooted in Bayesian updating and equilibrium concepts in financial economics, its implementation may not be as straightforward.

In particular, we have provided both theoretical and empirical results to shed light on how the straight application of the BL framework in active investment management can lead to unintended trades and risk taking, which in turn leads to a more risky portfolio than desired. Focusing on the mismatch between the Sharpe ratio optimization behind the BL framework and the information ratio optimization in the active investment industry, we propose a remedy that leads to portable alpha implementation of active portfolios. These resulting active portfolios reflect intentional and true investment insights of the investment process.

ENDNOTE

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[1]Formally, reverse SR optimization problem sets Π that solves $0 = \text{argmax}_{\varpi_a} (\varpi_a + \varpi_B)'\Pi - \lambda(\varpi_a + \varpi_B)'\Sigma(\varpi_a + \varpi_B)$. The reverse optimization solution in Equation (7) is valid only if none of the constraints, if any, is binding in determining the benchmark portfolio. Based on our anecdotal observations, many others ignore this important point in attempting to determine implied expected returns given a set of portfolio weights. For example, in Jones, Lim, and Zangari [2007], the authors derived individual stock alphas by reverse optimization of what they labeled as the optimal tile portfolio (OTP) (see their Equation (6)). Note that the reverse optimized alphas are valid only if none of the constraints is binding. However, the OTP is a result of constrained optimization according to their Equation (5). As a result, the derived alphas are distorted to an unknown extent, which may lead to suboptimal active portfolios relative to the original information content embedded in the OTP.

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