

RETURN FORECASTS AND OPTIMAL PORTFOLIO CONSTRUCTION: A QUANTILE REGRESSION APPROACH

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ABSTRACT. In finance, there is growing interest in quantile regression with the particular focus on value at risk and copula models. In this paper, we first present a general interpretation of quantile regression in the financial market. We then explore the full distributional impact of factors on returns of securities, and find that factor effects vary substantially across quantiles of returns. Utilizing distributional information from quantile regression models, we propose two general methods for the return forecasting and portfolio construction. We show that under mild conditions these new methods provide more accurate forecasts and potentially more value-added portfolios than the classical conditional mean method.

1. INTRODUCTION

There is intensive research on the return forecasts for securities, and most of it has focused on the conditional mean estimation strategy. The classical least squares methods and maximum likelihood estimator provide attractive methods of estimation for Gaussian linear equation models with additive errors. However, these methods offer only a conditional mean view of the causal relationship, implicitly imposing quite restrictive location-shift assumptions on the way that covariates are allowed to influence the conditional distributions of the response variables. Quantile regression methods seek to broaden this view, offering a more complete characterization of the stochastic relationship among variables and providing more robust, and consequently more efficient, estimates in some non-Gaussian settings.

Since the seminal paper of Koenker and Bassett (1978), quantile regression has gradually become a complimentary approach for the traditional conditional mean estimation methods. However, in the financial markets, quantile regression has not been employed until quite recently. Most of applications of quantile regression in finance have been focused on conditional value at risk models. Engle and Manganelli (2002) have elegantly laid out the natural interpretation of VaR models as the concept of quantile regression and considered the application of quantile regression to a nonlinear autoregressive VaR model. Giacomini and Komunjer (2002) proposed a Wald-type test to compare different forecasts resulted from the quantile regression

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for VaR models. Chen and Chen (2003) carried out some empirical comparisons and found that VaR calculations with the quantile regression approach outperform those with the variance-covariance approach, and the advantages of the quantile regression approach are more obvious in calculating VaRs if the holding periods are longer. In addition to the application of quantile regression to VaR models, Bassett and Chen (2001) analyzed the portfolio management styles based on the attribution of returns using a quantile regression approach. Barnes and Hughes (2002) applied the quantile regression to explore efficacy of the CAPM hypothesis over the distribution of returns. Besides the application of quantile regression in equity models, another line of the application is in the fixed income area, where binary quantile regression is employed to predict the default probability. For the theoretical exploration of binary quantile regression and applications in the fixed income markets, see Kordas (2002) and Kordas (2004).

While the current literature of quantile regression in finance has focused on risk (VaR model) or ex-post analysis of return, in this paper we focus on the return side along the line of forecasting and portfolio construction. We first provide a general interpretation of the quantile regression for not only the risk side but the return side as well. We then employ this new statistical methodology to explore the distributional effects of factors on the response of equity models. We introduce two general strategies to illustrate how the distributional information can be utilized to construct an optimal portfolio and show that the portfolio resulting from the distributional estimation method outperforms the portfolio constructed based on the classical conditional mean estimation methods.

Section 2 presents a brief review of equity models with an emphasis on the application of quantile regression methods to equity models. Section 3 describes the quantile regression method through a detailed algorithm and examples. Section 4 introduces a general interpretation of quantile regression in equity models as well as advantages and challenges compared to the traditional methods. Section 5 proposes two approaches for the construction of optimal portfolio using the distributional information. Section 6 concludes.

2. LITERATURE REVIEW

One question has remained central in financial markets: “Is it possible to forecast the return of securities?” Understanding of the financial markets has been advanced by numerous authors including (but not limited to) Sharpe (1964, CAPM), Fama and French (1992, three-factor model), Lakonishok (1988, 1992, behaviour finance). Through the advance of statistic developments and new technologies, the ability to forecast returns has been greatly enhanced. A typical quantitative approach for return forecasting is to build a linear or non linear model based on a combination of valuation, technical, and expectational factors and apply various conditional mean versions such as least squares methods for the estimation.

Indeed, applying the standard conditional mean estimation strategy to a multi-factor model across a diverse range of stocks is a popular approach to ascertain securities' return appeal. However, by treating different securities as the same, the above approach forces returns of all securities to have the homogeneous mean-reversion behavior. This mean treatment effect (MTE) view of the factor effect is valid under the extremely strong condition, i.e., the *average* marginal effect of a factor does not vary across the size of the factor and returns. However, we know that different securities may have different response for the exposure to the same factor. Thus, the MTE view is apparently not able to capture such characteristics of factor effects.¹ Realizing that the factor effects are not constant across securities, some studies such as Barnes and Hughes (2002) have pointed out that an estimate of a distributional effect would be preferred. Quantile regression is a natural tool to accomplish such a task.

A primary example of the growing interest for quantile regression in finance is in the context of risk management, as witnessed by the literature on Value at Risk. This is not a coincidence. Since VaR is simply a particular quantile of future portfolio values conditional on current information, the quantile regression is a natural tool to tackle such a problem. Engle and Manganelli (1999) were among the first to consider the quantile regression for the VaR model. They construct a conditional autoregressive value at risk model (CAViaR), and employ quantile regression for the estimation. To evaluate the goodness of fit for the estimated results, they also propose a test based on the idea in Chernozhukov (1999). Their applications to real data suggested that the tails follow different behavior from the middle of the distribution, which contradicts the assumption behind GARCH and RiskMetrics since such approaches implicitly assume that the tails follow the same process as the rest of the returns. Following Engle and Manganelli (1999), Giacomini and Komunjer (2002) propose a testing procedure for the analysis of competing conditional quantile regression forecasts proposed in the literature for the VaR model. The test is based on the weight assignments for results from different methods, which is parallel to the encompassing principle in the context of conditional mean forecasting.²

Recently, Chen and Chen (2003) carried out an empirical study to compare performances of Nikkei 225 VaR calculations with the quantile regression approach to those with the conventional variance-covariance approach. They find that VaR calculations with quantile regression outperform those with the variance-covariance approach. Furthermore, the advantage of the quantile regression is more significant for longer holding periods. While the empirical results are encouraging, the application to other equity markets would be very interesting.

In addition to the application in VaR models, there are studies employing quantile regression in other areas of financial market. Barnes and Hughes (2002) employ

¹By specifying certain function forms for a factor, the MTE view may reveal some variations of factor effects across different values of factors.

²For the discussion of the encompassing principle in the context of conditional mean forecasting, please see Hendry and Richard (1982), Mizon and Richard (1986) and Diebold (1989).

quantile regression to test whether the conditional CAPM holds at points of the return distribution other than the mean. The empirical results of Barnes and Hughes (2002) provide new support for Merton's (1987) prediction that size and returns are positively related which is in contrast to many of the empirical findings reported in the literature. Realizing that it would be a mistake to think a single measure should describe a portfolio's style, Bassett and Chen (2001) employ quantile regression to characterize the distributional view of portfolio management styles. They find that in the small cap value estimates, large positive and negative impacts in the tails cancel each other which leaves barely the hint of the conditional mean result.

While the use of quantile regression in various financial fields has enriched the understanding of financial market, one great potential would be to apply this new tool to the quantitative investment practice. To the best of our knowledge, there has been no study employing quantile regression for the purpose of return forecast and portfolio construction. As an initial effort on this topic, this paper explores some basic methods which might be used to link the quantile regression to the portfolio construction. We try not to be too specific about the formulation of the model and properties of the results, but rather provide a general view on how quantile regression could be used for better return forecasting and hence portfolio construction.

3. QUANTILE REGRESSION: A BRIEF INTRODUCTION

Why do we need quantile regression? In most cases, the effects of factors on response are not constant, but rather vary across the responses. Suppose we would like to know: How does the debt-ratio factor affect a stock's return at high and low values? By using the conditional mean estimation method, we would come to the result that in spite of different return levels, the debt ratio will affect the returns of all securities exactly the *same* way. However, if there is heterogeneity in the effects, then different strategies are needed to handle such differences.

How do we capture such heterogeneous causal effects? To answer this question, consider the following simple linear model:

$$(3.1) \quad y_{i,t} = x_{i,t}^\top \beta + u_{i,t},$$

where $y_{i,t}$ is the return and $x_{i,t}$ is the k -dimension vector of factors of the stock i at time t , $i = 1, 2, \dots, N$, $t = 1, 2, \dots, T$. Assume x is independent of u . For simplicity, consider only the cross-section version of (3.1), i.e., t is fixed at only one period. For notation convenience, we will hereafter drop the reference to the index t unless it is needed for clarification.

For the pure location shift model (3.1), under the conditions that u_i is i.i.d. and Gaussian, the traditional conditional mean estimation strategy would yield consistent and efficient estimates for the constant causal effect of x on y . However, such restrictive conditions of i.i.d. and Gaussian errors rarely hold in market data. This model ignores the distributional view on how x effects y , or in other words, heterogeneity. It

is usually true that the effect of x on y would not be constant across y . For example,

$$(3.2) \quad y_i = x_i^\top \beta + \phi(x_i, \gamma) \epsilon_i;$$

where $\phi(\cdot)$ is usually unknown. Note that now $u_i = \phi(x_i, \gamma) \epsilon_i$ is not i.i.d. anymore even if ϵ_i is assumed to be i.i.d. How does the location-scale shift model, (3.2), capture the causal effects of x on y distributionally?

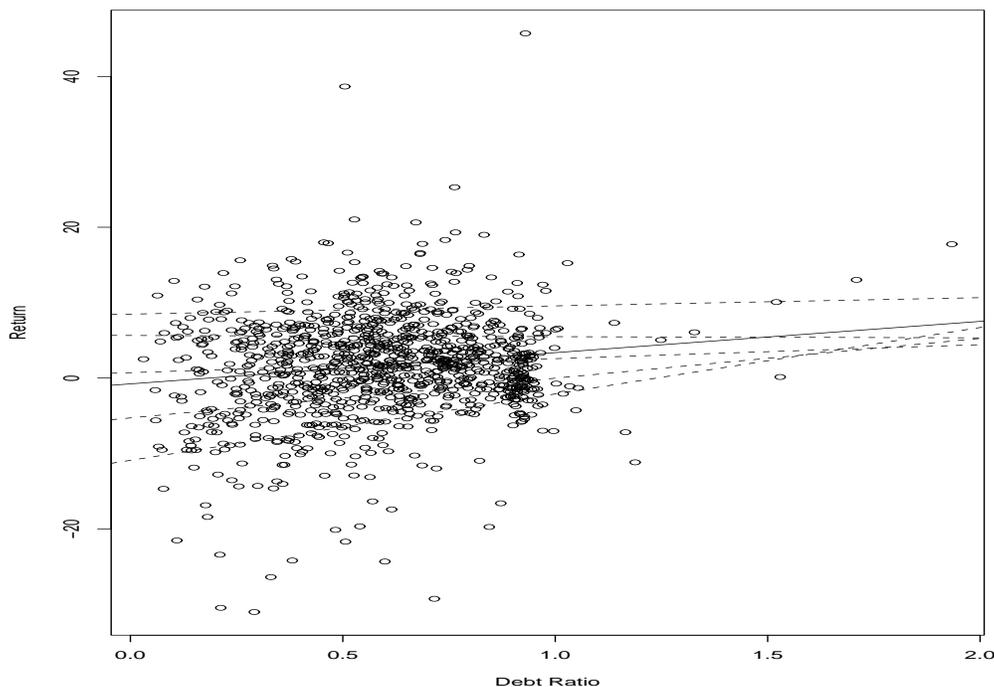


FIGURE 3.1. Heterogenous Effects of Debt Ratio on Return. The data is from the May, 2004 observations of all equities included in the ValueLine database. The solid line is the OLS result while the dashed lines are the quantile regression results at $\tau = \{0.1, 0.25, 0.5, 0.75, 0.9\}$. The different slopes of quantile regression lines indicate that the debt ratio has quite different impacts on the returns at different levels.

The quantile regression approach provides the answer. For the purpose of illustration, consider the simplest single factor version of (3.2),

$$(3.3) \quad y_i = \beta_0 + \beta_1 x_i + (x_i \gamma) \epsilon_i.$$

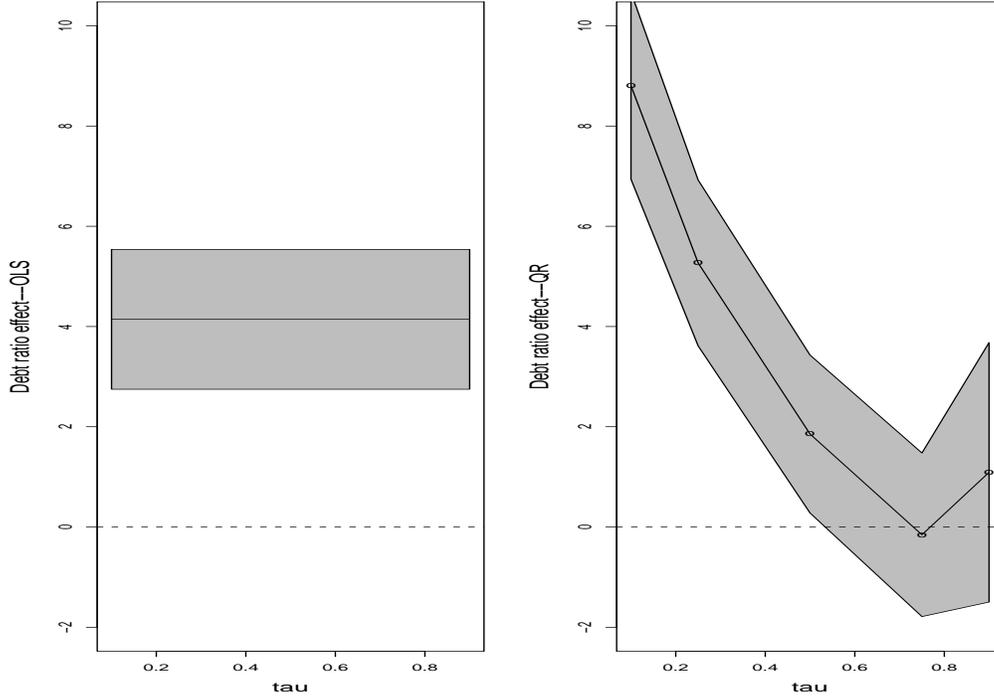


FIGURE 3.2. Heterogeneous Effects of Debt Ratio on Return. The OLS and QR results are shown in the left and right plot, respectively. The five points in the QR plot are the slope estimates at $\tau = \{0.1, 0.25, 0.5, 0.75, 0.9\}$ and the grey area indicates the 90% confidence band.

Say, x_i is the debt ratio. For (3.3), assuming ϵ_i is independently distributed with the distribution function F_i , we have

$$\begin{aligned}
 (3.4) \quad F_{y_i}^{-1}(\tau|x_i) &= \beta_0 + \beta_1 x_i + (x_i \gamma) F_i^{-1}(\tau) \\
 &= \beta_0 + (\beta_1 + \gamma F_i^{-1}(\tau)) x_i
 \end{aligned}$$

where $\tau \in (0, 1)$. Correspondingly, the conditional quantile regression model is

$$(3.5) \quad Q_{y_i}(\tau|x_i) = \beta_0(\tau) + \beta_1(\tau)x_i,$$

where $\beta_0(\tau) = \beta_0$ and $\beta_1(\tau) = \beta_1 + \gamma F_i^{-1}(\tau)$. Thus, with τ varying within the range $(0, 1)$, model (3.5) presents a distributional view of the effect of debt ratio on return.

It should be noted that the conditional mean value, $\frac{\partial E(y|x)}{\partial x}$, usually referred to as the mean treatment effect (MTE), could be derived from the quantile value, referred

to as the quantile treatment effect (QTE),

$$MTE = \int_0^1 \beta(t) dt = \beta_1 + \gamma E(\epsilon) = \beta_1 + u_\epsilon.$$

In particular, if ϵ is mean zero, then MTE is the classical conditional mean value, β_1 . This relationship between MTE and QTE suggests that QTE is the decomposition of MTE and thus provides a full picture view on the causal effects. Note that the MTE in models with pure location shift is exactly what would be estimated by the least squares method.

The above point is illustrated vividly in Figure 3.1–3.2, where the effects of debt ratio on returns are explored by using both the OLS and QR methods. Figure 3.1, which is a scatter plot with slopes obtained from OLS and QR at $\tau = \{0.1, 0.25, 0.5, 0.75, 0.9\}$, shows that there is dramatic variation of the effects of debt ratio across quantiles of conditional return distribution. The magnitude of heterogeneity can be seen more clearly in Figure 3.2. While both OLS and QR results indicate that increase of debt ratio will benefit return, the QR tells a more detailed story on how debt ratio affects return. The left plot of the OLS result implies that the debt ratio has a uniformly significant marginal effect of 4.1 on the return. The right plot of the QR results suggests that such causal effects are not at all constant: the effect is as high as 9 for the companies with returns in left tail and becomes insignificant as the conditional quantile of return passes the median! Although this is just a one-factor model, the intuition is clear that the effects of debt ratio are heterogenous across returns.

The parameter $\beta(\tau) = (\beta_0(\tau), \beta_1(\tau))^T$ in model (3.5) can be estimated by solving the following conditional quantile objective function:

$$(3.6) \quad \min_{b \in \mathcal{B}} \sum_{i=1}^n \rho_\tau(y_i - \tilde{x}_i^T b),$$

where $\tilde{x}_i = (1, x_i)$ and the check function $\rho(\cdot)$ is defined as $\rho_t(e) = (t - I(e \leq 0))e$.³

Quantile regression has been well developed for the single linear equation model, both in terms of estimation and inference. Recently, the research on QR has been furthered and broadened greatly in many areas, such as, discrete response models (Kordas, 2004), panel data models (Koenker, 2004), time series models (Koenker and Zhao, 1996, Koenker and Xiao, 2003), survival models (Powell, 1986, Koenker and Oga, 2002, Portnoy, 2004), and structural equation models (Amemiya, 1982, Chesher, 2003, Ma and Koenker, 2004, Imbens and Newey, 2003).

³The check function is the short notation for the objective function:

$$\sum_{e_i \geq 0} \tau e_i + \sum_{e_i < 0} (\tau - 1)e_i.$$

4. A GENERAL INTERPRETATION OF QR IN EQUITY MODELS

In this section, we first present a very general interpretation of QR in equity models and then illustrate the interpretation in detail through a popular multi-factor model. We discuss the advantages of QR over classic methods and challenges of using the distribution information for return forecasting and portfolio construction.

4.1. A General Interpretation. Consider the semiparametric model with a general form,

$$(4.1) \quad y = G(x, \beta, u),$$

where y is the return of a security, x is the k -dimension vector of factors, u is the error term, which is assumed to be i.i.d. with distribution function F . Note that x may include the lagged response variables. We assume that x is independent of u . The function $G(\cdot)$ is rather flexible to make the function either linear or nonlinear.

Under some mild conditions, we can write the conditional quantile functions,⁴

$$(4.2) \quad \begin{aligned} Q_y(\tau|x) &= G(x, \beta, F_u^{-1}(\tau)) \\ &= G(x, \beta(\tau)). \end{aligned}$$

How do we interpret (4.2) in financial markets? One approach is to regard (4.2) as a natural model for value at risk, which is defined as the value that a portfolio will lose with a given probability (τ) over a certain time. While this is an excellent interpretation, we think model (4.2) might tell far more than just VaR. Since risk and returns are just two sides of the same coin, model (4.2) could be interpreted as the value at risk, or the return depending on the emphasis. Without any context, model (4.2) tells that the probability of return at τ th percentile conditional on x is $\beta(\tau)$. In equity market, $\beta(\tau)$ tells how the information contained in x affect the returns at different points of distribution. Thus, model (4.2) presents a complete view of causal effects or forecasting power of the selected factors on returns.

There are several immediate implications of our general interpretation. First of all, the QR is a general alternative approach for quantitative analysis of the financial market. Secondly, VaR models are just a special case where τ is at extreme values. While VaR models focus on the interpretation of τ in (4.2), the return forecasting models focus on character of the heterogeneity of the causal effects, $\beta(\tau)$. The general

⁴The detailed conditions for (4.1) are specified in more detail in the following:

A.1: The conditional distribution functions $F_{y_i}(y_i|x_i)$ is absolutely continuous with continuous densities f_i that is uniformly bounded away from 0 and ∞ at the points $\xi_i = Q_{y_i}(\tau|x_i)$ for $i = 1, \dots, n$.

A.2: The function $g(\cdot)$, is assumed strictly monotonic in u , and differentiable with respect to x .

Note that the above conditions are rather standard in the QR literature. For more discussion, see Koenker (2005).

interpretation expands the scope of application of QR to areas beyond VaR models.⁵ Thirdly, QR does not require the Gaussian and i.i.d. errors which are the conditions of traditional conditional mean methods, and thus allow us to explore the heterogeneous effects of factors on the whole distribution of returns.

4.2. An Empirical Example. In this example we use monthly data from Valueline for the US equity market from January 1990 to June 2004.⁶ We focus on the large cap market, which consists of stocks in the Russell1000, S&P500 and S&P400 indices for about 1,100 securities each month. We use the one-month return as the dependent variable and the security's value, technical and expected information as the independent variables. The value regressors include the book to price (BTOP), earnings to price (ETOP), debt ratio (Dratio), retained earnings to assets (REOA), liabilities to price (LTOP) and zscore (Zscore). Note that earnings are measured by operating income before depreciation and amortization. Liabilities in LTOP are the total reported liabilities. Zscore is the linear combination of several factors to measure the firm's financial strength.⁷ The debt ratio is calculated as total liabilities divided by total assets. At the technical level, the variables include a reversal factor measured by 3-month lag return (Ret_L3) and market cap (VL_CAP). At the expected information level, we control for earnings per share of the next year from IBES (IBES_EPS).

A brief statistical summary of the above mentioned variables is reported in Table B.1. The quantiles, means and standard deviations for the variables illustrate a number of interesting characteristics of the sample. The first interesting fact is that, the variation of all variables except for ETOP and Dratio are very high, indicating that identification would not be a problem. Even for the factors of BTOP and Dratio, the range is very high and both factors have symmetric distributions. Another interesting feature of the sample at the value-factor level is that, there is a wide range of Zscore from -39.75 to 127 with an average of 4.15. For the technical factors, note that there is a considerable variation of firm sizes measured by market cap, and the mean is 3 times the median suggesting a large skewness toward the right tail. At the expectational-factor level, the most interesting fact is that all the forecasts of one-year forward IBES_EPS are positive! Since the sample is from January 1990 to July 2004 for the large cap universe, we suspect the credibility of this factor. We think that for the estimates of EPS from the analysts' point, a negative value might not be a good thing to report. Therefore, there appears to be a moral screening process for this factor.

⁵It should be noted that the density is difficult to estimate as $\tau \rightarrow 0$ and $\tau \rightarrow 1$.

⁶The financial ratios are calculated based on the following rules: use the quarterly data first but if the quarterly data is missing then use the annual data.

⁷In this paper, Zscore is calculated as follows: $Zscore = 3.3ROA + 0.999SOA + 0.6EOD + 1.2WKOA$, where ROA is return on total assets, SOA is sales on assets, EOD is equity on debt, $WKOA$ is working capital on assets. Note that this is a modified version of the Altman Zscore defined in Altman (1968).

We employ the following simple linear quantile regression model,

$$(4.3) \quad Q_y(\tau|x_1, x_2, x_3) = x_1^\top \beta_1(\tau) + x_2^\top \beta_2(\tau) + x_3^\top \beta_3(\tau),$$

where x_1 are the value factors including BTOP, ETOP, Dratio, REOA, LTOP, and Zscore; x_2 are technical factors which controls for Ret_L3 and VL_CAP; x_3 represents the expectational factors from IBES, IBES_EPS.

Model (4.3) is estimated at $\tau = \{0.1, 0.25, 0.3, 0.4, 0.5, 0.6, 0.75, 0.8, 0.9\}$. The results of parameter estimates are depicted in Figure 4.1 with 90% confidence band. As the benchmark for comparison, OLS results are also plotted as bold lines in Figure 4.1. An overall impression suggests that QR reveals significant and interesting results that are hidden in the traditional OLS model.

Consider first the effects of value factors on the return. For the BTOP factor, the effect is not significant at the left tail of the return distribution but as τ moves upward, the effect increases and is about 4 at $\tau = 0.8$! Note that the OLS result is close to the median, which is barely significant. For the debt ratio, the pattern of effects is just the opposite of that of BTOP: the increase of Dratio has positive effects on the companies having lower returns; as return increases, the effects decrease and become insignificant after the median. Although we expect that ETOP should behave similarly to BTOP, we find that ETOP is insignificant except at the left tail. For the factor of REOA, the effects are insignificant at the median, but very significant at the tails, meaning that the increase of REOA has a very positive effect on lower returns and negative effects on high returns. For LTOP, we find that the increase of LTOP will further reduce the value of the lower-return companies but benefit the higher-return ones. The Zscore reflects the financial strength, which is expected to have positive effects on the returns. The increase of Zscore is effective only at right tails (it is barely significant at the left tail). The higher the Zscore, the larger the impact on the high-return stocks. For OLS, the Zscore is not significant at all.

At the technical level, we find that the reversal factor does work as a reversal but only for the low return stocks: for high return stocks, the prior returns do not matter, but for the low returns, the history of returns does have negative effects. Note that OLS is the same as the zero line. We find that size does matter, but in a very heterogenous way: the increase of size has positive effect on lower return stocks but negative effect on the high return stocks.

Finally, for the effects of analysts forecasts, IBES_EPS, we find that the one year forecast of the earnings-per-share has a negative effect on the stocks with lower returns but no effect on the stocks with high returns. This finding indicates that betting against the analysts might not be a bad strategy.

Note that for most factors, the results from OLS are not significant. The comparison of QR results with OLS results suggests that one should interpret findings of insignificant mean effects with considerable caution since it appears that those results arise from averaging significant benefits from reductions in factor values for

high return stocks and significant benefits from increase in factor values for low return stocks.

4.3. Advantages and Challenges. It is clear that there are advantages of quantile regression over the traditional ones such as least squares. First of all, instead of the point estimate for the conditional mean, we have the whole distribution. With τ varying in the range of $(0, 1)$ we potentially have different $\beta(\tau)$. Therefore we would know not only the expected average return, but the whole distribution of the expected return in the next period given the information known at the current moment. Secondly, as shown earlier, the conditional mean result could be derived from the conditional quantile effect. In particular if the distribution of causal effects are not too skewed, the conditional mean effect would be close to the median. Therefore, for the same sample, quantile regression reveals more information than classical methods.

For our return forecasting and final portfolio construction, the main concern is how the quantile regression method can be used to yield more reasonable forecasts of returns and hence the optimal portfolio. For the classical approach, we have only one set of point estimates, and then one forecast for each stock for the asset return. For example, for the conditional mean approach, we only have $\hat{E}(y_{i,t+1}|x_{i,t})$ for each stock i . But now given multiple sets of forecasted returns, say, for example, at the conventional quantiles,

$$\tau = \{0.1, 0.25, 0.5, 0.75, 0.9\},$$

the forecasts of return will be

$$\{\hat{Q}_{y_i}(0.1|x_i), \hat{Q}_{y_i}(0.25|x_i), \hat{Q}_{y_i}(0.5|x_i), \hat{Q}_{y_i}(0.75|x_i), \hat{Q}_{y_i}(0.9|x_i)\}.$$

How can we take advantage of such distribution information to have more accurate forecasts and more value-added portfolio? To the best of our knowledge for quantitative analysis of equity models, this is the first study to address and investigate such an issue. We explore the answers in the next section.

5. RETURN FORECAST AND PORTFOLIO CONSTRUCTION USING DISTRIBUTIONAL INFORMATION

Using the distributional information from the quantile regression estimation strategy, we propose two methods to forecast returns and hence construct an optimal portfolio: quantile regression alpha distribution (QRAD) and quantile regression portfolio distribution (QRPD). Before going into details of QRAD and QRPD, we introduce some definitions and propositions to measure the goodness-of-forecast.

Definition 1. Suppose $R_{i,t}$ is the return of stock i at time t and $\hat{R}_{i,t}$ is the forecasted return for the same stock. Then an accuracy measurement is defined as $\sum_{i=1}^N |\hat{R}_{i,t} - R_{i,t}|$.

We believe that the sum of absolute values is a better measurement of accuracy than the more usual sum of quadratic values for the return of security. The absolute value is robust compared to the quadratic one and the absolute value measure closeness better than the quadratic term does. Under the special case of symmetric distribution of the returns, the two measurements will be identical.

Definition 2. Suppose $R_{i,t}$ is the return of stock i at time t , $\hat{R}_{i,t}$ is the forecasted return for the same stock, then the forecast method is said to be optimal at time t iff it yields $\hat{R}_{i,t}$ minimizing $\sum_{i=1}^N |\hat{R}_{i,t} - R_{i,t}|$.

A natural question would be: How to find such optimal forecasting results to achieve the desirable degree of accuracy?

Proposition 1. Suppose R^m is the median return and R^e is the expected return of a set of returns R_i , $i = 1, 2, \dots, N$, then

$$\sum_{i=1}^N |R^m - R_i| \leq \sum_{i=1}^N |R^e - R_i|.$$

Proof: The proof follows immediately from the definition of median and expected return. ■

Now consider extending the unconditional median (expected) return to the conditional ones,

$$(5.1) \quad R_{i,t+1} = G(x_{it}, \beta, u_{i,t+1})$$

and let $\hat{R}_i^m(0.5|x)$ and $\hat{R}_i^e(\cdot|x)$ denote the forecasted returns from the conditional median and mean OLS estimate, respectively. The above proposition still holds.

Proposition 2. Let $\hat{R}_i^m(0.5|x)$ be the median return and $\hat{R}_i^e(\cdot|x)$ be the expected return from (5.1), then

$$\sum_{i=1}^N |\hat{R}_i^m - R_i| \leq \sum_{i=1}^N |\hat{R}_i^e - R_i|.$$

Proof: The proof is straightforward and it comes directly from the minimization of the following objective function:

$$\hat{\beta}(0.5) = \operatorname{argmin}_{b \in B} \sum_{i=1}^N \rho_\tau(R_{i,t+1} - x_{i,t}^\top b) = \operatorname{argmin}_{b \in B} \sum_{i=1}^N |R_{i,t+1} - x_{i,t}^\top b|,$$

$$\hat{R}^m(0.5|x) = x^\top \hat{\beta}(0.5).$$

So far, for (5.1), only the median estimates are used. What if we have the distributional estimates? Is there a way that a combination of estimates from the entire distribution would be better than the median given that the median assumes that all next-period returns are in the line of median of last period? ■

Consider again model (5.1). For the simplicity, let $\tau = (0.1, 0.5, 0.9)$, then the corresponding forecasted returns will be⁸

$$\hat{R}_{0.1} = x^\top \hat{\beta}(0.1), R_{0.5} = x^\top \hat{\beta}(0.5), R_{0.9} = x^\top \hat{\beta}(0.9).$$

However, for stock i , we do not know the specific quantile of the next return since the return is not known. So which $\hat{\beta}(\tau)$ should be used and how accurate would be the result? A second question is, do better forecasts measured by the accuracy in Definition 1 imply more value-added portfolio? We'll explore answers to these questions in following subsections.

5.1. The QRAD Method. We introduce two sub approaches for the QRAD method, namely, QRAD Location and QRAD Probability. We will discuss each of them in detail in the following.

5.1.1. QRAD Location. The QRAD Location method is as follows. According to the last return, the conditional distribution is derived and returns are grouped by quantiles. Corresponding to the quantile each stock belongs to, we could assign the $\beta(\tau)$ for the forecast of next period. The validity of this strategy is based on the logic that the rank of the return of stocks does not change dramatically for two continuous periods. This condition imposes a restriction for application but we believe that such a condition is less restrictive than the assumptions made implicitly for the conditional median and mean case, where all of the next period returns are assumed to be on the same line of the mean or median of the previous period. Thus by the QRAD-Location method, we have the forecast for blocks of stocks which are in the range of the same quantile. The final forecasted results for all stocks in the sample would be the combination of forecasts for these blocks.

For example, if $\tau = \{0.1, 0.5, 0.9\}$ is employed, then at the time t , we have the conditional quantile for $R_{i,t}$, say, it belongs to $(0, 0.1]$, then we would use $\hat{\beta}(0.1)$ to get the next period forecast $\hat{R}_{i,t+1} = x^\top \hat{\beta}(0.1)$. Applying the same rule, we would obtain the forecast returns for all stocks in the sample,

$$(5.2) \quad \hat{R}_{i,t+1} = \begin{cases} x_{i,t+1}^\top \hat{\beta}(0.9) & \text{if } R_{i,t} \geq x_{i,t}^\top \hat{\beta}(0.9) \\ x_{i,t+1}^\top \hat{\beta}(0.1) & \text{if } R_{i,t} \leq x_{i,t}^\top \hat{\beta}(0.1) \\ x_{i,t+1}^\top \hat{\beta}(0.5) & \text{otherwise} \end{cases} .$$

The goodness-of-forecast might be decomposed as follows,

$$\sum_{i=1}^N |\hat{R}_{i,t+1} - R_{i,t+1}| = \sum_A |\hat{R}_{i,t+1} - R_{i,t+1}| + \sum_B |\hat{R}_{i,t+1} - R_{i,t+1}| + \sum_C |\hat{R}_{i,t+1} - R_{i,t+1}|,$$

⁸Note that we drop the subscript for time for the simple notation.

where A , B and C denotes the area of $(0, 0.1]$, $(0.1, 0.9)$ and $[0.9, 1)$, respectively, for the quantile location of $R_{i,t}$ (See Figure 5.1). The relative accuracy of the forecasts based on the QRAD Location method is stated in the following proposition.

Proposition 3. *Let $\hat{R}^l(\tau|x)$ be the composite quantile return from the QRAD Location method, also $\hat{R}^m(0.5|x)$ and $\hat{R}^e(\cdot|x)$ be the median and mean return from (5.1), respectively, then*

$$\sum_{i=1}^N |\hat{R}_{i,t+1}^l - R_{i,t+1}| \leq \sum_{i=1}^N |\hat{R}_{i,t+1}^m - R_{i,t+1}| \leq \sum_{i=1}^N |\hat{R}_{i,t+1}^e - R_{i,t+1}|,$$

where the letter l , m and e denotes location, median and mean, respectively.

Proof: We need the proof only for the first inequality. Let $\Delta_1 \equiv |\hat{R}_{i,t}^l - R_{i,t}|$ and $\Delta_2 \equiv \sum_{i=1}^N |\hat{R}_{i,t}^m - R_{i,t}|$. The decomposition by location yields

$$\begin{aligned} \Delta_1 &= \sum_A |\hat{R}_{i,t+1}^{0.1} - R_{i,t+1}| + \sum_B |\hat{R}_{i,t+1}^m - R_{i,t+1}| + \sum_C |\hat{R}_{i,t+1}^{0.9} - R_{i,t+1}|, \\ \Delta_2 &= \sum_A |\hat{R}_{i,t+1}^m - R_{i,t+1}| + \sum_B |\hat{R}_{i,t+1}^m - R_{i,t+1}| + \sum_C |\hat{R}_{i,t+1}^m - R_{i,t+1}|. \end{aligned}$$

Then, $\Delta_1 \leq \Delta_2$ is equivalent to

$$\sum_A |\hat{R}_{i,t+1}^{0.1} - R_{i,t+1}| + \sum_C |\hat{R}_{i,t+1}^{0.9} - R_{i,t+1}| \leq \sum_A |\hat{R}_{i,t+1}^m - R_{i,t+1}| + \sum_C |\hat{R}_{i,t+1}^m - R_{i,t+1}|.$$

By the condition that for stocks with $R_{i,t} \in A$,

$$Prob.(R_{i,t+1} \in A) \geq Prob.(R_{i,t+1} \in B) \text{ and } Prob.(R_{i,t+1} \in A) \geq Prob.(R_{i,t+1} \in C),$$

we have

$$\sum_A |\hat{R}_{i,t+1}^{0.1} - R_{i,t+1}| \leq \sum_A |\hat{R}_{i,t+1}^m - R_{i,t+1}|.$$

The same inequality holds for C . The final result follows immediately from the combination. \blacksquare

The results of Proposition 3 can be extended to situations with more than three areas. However, it should be noted that there is the trade off between the number of divisions and the accuracy of forecast.

5.1.2. QRAD Probability. To overcome the disadvantage of the QRAD Location method where forecasts depend heavily on previous conditional location, we propose an alternative approach, which is to assign probabilities to the forecasts,

$$\hat{R}_{i,t+1}^p = p_1 \hat{R}_{1,i,t+1} + p_2 \hat{R}_{2,i,t+1} + \dots + p_k \hat{R}_{k,i,t+1},$$

where p_k is the probability of the occurrence of $\hat{R}_{k,i,t+1}$. For example, for $\tau = \{0.1, 0.5, 0.9\}$, we would have that

$$(p_1, p_2, p_3) = (0.1, 0.8, 0.1),$$

and,

$$(\hat{R}_{1,i,t+1}, \hat{R}_{2,i,t+1}, \hat{R}_{3,i,t+1}) = x_{i,t}^\top (\hat{\beta}(0.1), \hat{\beta}(0.5), \hat{\beta}(0.9)).$$

Clearly, the above formulation is the familiar expected value. However, it should be noted that the expected forecasted return is not identical to the forecast of expected return in general case, that is, $E\hat{R} \neq \widehat{ER}$.

It can be shown that,

$$\sum_{i=1}^N |\hat{R}_{i,t+1}^p - R_{i,t+1}| \leq \sum_{i=1}^N |\hat{R}_{i,t+1}^m - R_{i,t+1}| \leq \sum_{i=1}^N |\hat{R}_{i,t+1}^e - R_{i,t+1}|.$$

Under mild conditions, both QRAD Location and QRAD Probability yield better goodness-of-forecast than the traditional methods. What is the relationship between QRAD Location and QRAD Probability? We find that $\hat{R}_{i,t+1}^p = E\hat{R}_{i,t+1}^l$. Note that from (5.2), under the conditions in Proposition 3 and by the definition of quantile regression, the probability chart (Figure 5.2) follows. In other words, the two methods are asymptotically the same. Thus we have used more information to construct the forecast and these forecasted results are more accurate than the conditional mean forecast which does not differentiate the factor effects for different level of returns.

5.2. The QRPD Method. The QRPD method differs from the QRAD method in that we use the distributional information at the optimization stage.

Corresponding to the forecasted returns at each conditional quantile, the optimal portfolio can be constructed. Let $W_\tau = (w_{1,\tau}, \dots, w_{N,\tau})^\top$ be the optimal weights of the portfolio resulted from using the τ th quantile regression forecasted returns. Then with $\tau \in (0, 1)$, we have an empirical distribution of the portfolio at time t . A natural question is, how to carry out the actual portfolio selection with so many “optimal” choices? Using the same strategy as the QRAD-probability method, the final portfolio at time t might be constructed as follows

$$W = p_1 W_{\tau_1} + \dots + p_k W_{\tau_k},$$

where p_k is the probability of occurrence of W_{τ_k} . For example, as $\tau = \{0.1, 0.5, 0.9\}$, we would have three sets of weights, $W_{0.1}$, $W_{0.5}$ and $W_{0.9}$, corresponding to the three sets of forecasting returns. Thus, the weights of the proposed portfolio will be

$$W = 0.1W_{0.1} + 0.8W_{0.5} + 0.1W_{0.9}.$$

With this new portfolio construction methodology, there are two questions we need to answer. First of all, what is the relationship between QRAD and QRPD? Do they yield the same portfolio? Second, does the portfolio constructed from the QRPD method outperform those from the median or mean forecast? That is, if the performance is measured by the value added subject to the same constraints and the same benchmark, does the inequality, $W_q^\top R \geq W_m^\top R$, hold?

To explore the answer to the first question, consider the following standard objective function,

$$(5.3) \quad \max_W W^\top R - \lambda W^\top \Omega W,$$

where Ω is the covariance matrix of R , λ is the risk acceptance parameter. With the same value of λ , we derive the sufficient conditions that the methods of QRAD-Probability and QRPD yield the same portfolio. Assume that R_1 is independent of R_2 and that $\Omega_1 = \Omega_2$. Furthermore, suppose R_k is the forecasted returns from τ_k quantile regression, then by QRAD Probability, we have

$$R = p_1 R_1 + p_2 R_2.$$

The first order condition of (5.3) yields

$$\begin{aligned} W_{qrp} &= \frac{1}{2\lambda} \Omega^{-1} R \\ &= \frac{1}{2\lambda(p_1^2 + p_2^2)} (p_1 \Omega_1^{-1} R_1 + p_2 \Omega_2^{-1} R_2) \\ &= \frac{1}{p_1^2 + p_2^2} (p_1 W_1 + p_2 W_2) \\ &= W_{qradp}. \end{aligned}$$

The assumption of equality of the covariance matrix is too restrictive to be true in general. However, if the objective function is convex, then by Jensen's inequality, it is expected that the following inequality of portfolio value holds: $V_{QRAD-Probability} \geq V_{QRPD}$.

Regarding the second question, our answer is that it is not necessary although the probability is positive. This is simply because of the constraints imposed at the optimization stage. Since any forecast would not completely dominate another in the sense of point by point but rather in a law of large number sense, then the constraints might end up restricting the feasible set to an area where the better forecasts are not active. However, from the statistical perspective, it is expected that in general, the better forecasts would give us more chances to construct a better portfolio. The mathematical version of this argument is provided in the appendix.

6. CONCLUSION

The quantitative analysis for the equity return forecasting and hence portfolio construction is becoming more and more popular in both the academic and industrial world. The statistical methodology such as estimation of expected returns plays a central role in the quantitative analysis. However, most of the analytical approaches are based on the conditional mean method, which ignores the heterogeneity of the effects of factors on returns.

In this paper, we emphasize the heterogeneity issue from the response side and introduce quantile regression as a natural statistical tool to tackle such an issue. We present a general interpretation of the quantile regression results for equity models by expanding the interpretation to not only conditional risk but also the conditional return as well. Regarding quantile regression as a general alternative approach to the classical conditional mean method, we then focus on the return forecast and portfolio construction taking advantage of the distribution information from quantile regression. The main challenge is how to utilize the distribution information to construct more accurate forecast and better-performing portfolios given that the quantile of future return is unknown. To accomplish such tasks, we propose two methods, QRAD and QRPD, where the former utilizes the distributional information at the forecasting stage while the latter at the portfolio construction stage.

By using the goodness-of-forecast measurement, we show that results from both QRAD and QRPD outperform the results from traditional methods. Regarding the future research, we plan to employ these proposed methods to carry out an empirical study for US equity market.

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APPENDIX A. PROPOSITION 4

Proposition 4. *Suppose R_1 and R_2 are two sets of forecasting returns for the same set of stocks with real return R , and R_1 is better than R_2 by the goodness-of-forecast:*

$$(A.1) \quad |R_1 - R| \leq |R_2 - R|.$$

then under conditions that $\Omega_1 = \Omega_2 = \Omega$, where Ω_j is the covariance matrix, we have that with the same set of constraints Λ ,

$$(A.2) \quad W_1^\top R_1 \geq W_2^\top R_2,$$

where W_j is derived from

$$\max W_j^\top R_j - \lambda W_j^\top \Omega_j W_j, \quad s.t. \Lambda.$$

Proof: We'll prove the result in a reverse way. Suppose W is the weights for R for the constraint set Λ . Then we have

$$(A.3) \quad W^\top R \geq W_1^\top R \geq W_2^\top R.$$

By the condition that $W_j = g(\lambda, \Lambda) \Omega_j^{-1} R_j$, (A.3) is equivalent to

$$\begin{aligned} W^\top R &\geq R_1^\top \Omega_1^{-1} R \geq R_2^\top \Omega_2^{-1} R \\ \iff W^\top R &\geq W^\top R_1 \geq W^\top R_2 \\ \iff 0 &\leq W^\top R - W^\top R_1 \leq W^\top R - W^\top R_2 \\ \iff |W^\top R - W^\top R_1| &\leq |W^\top R - W^\top R_2| \\ \iff W^\top |R_1 - R| &\leq W^\top |R_2 - R| \end{aligned}$$

Since $W = \{w_i \geq 0\}$, i.e., each single weight in W is nonnegative, a strong sufficient condition for the last inequality is (A.1):

$$|R_{1i} - R_i| \leq |R_{2i} - R_i|.$$

■

Remarks

(i) Condition (A.1) is the strongest measure of goodness-of-forecast. A rather weaker condition is:

$$(A.4) \quad Prob(|R_{1i} - R_i| \leq |R_{2i} - R_i|) \geq 0.$$

(ii) However, condition (A.4) is still quite strong from the practical point of view. A rather general condition is

$$\sum |R_1 - R| \leq \sum |R_2 - R|,$$

which, although is not sufficient to yield the optimal portfolio, implies a higher chance.

(iii) In sum, better forecasts do not necessarily yield better portfolios.

APPENDIX B. TABLE FOR SUMMARY OF FACTOR STATISTICS

Tablepanasummary .5

.5.5

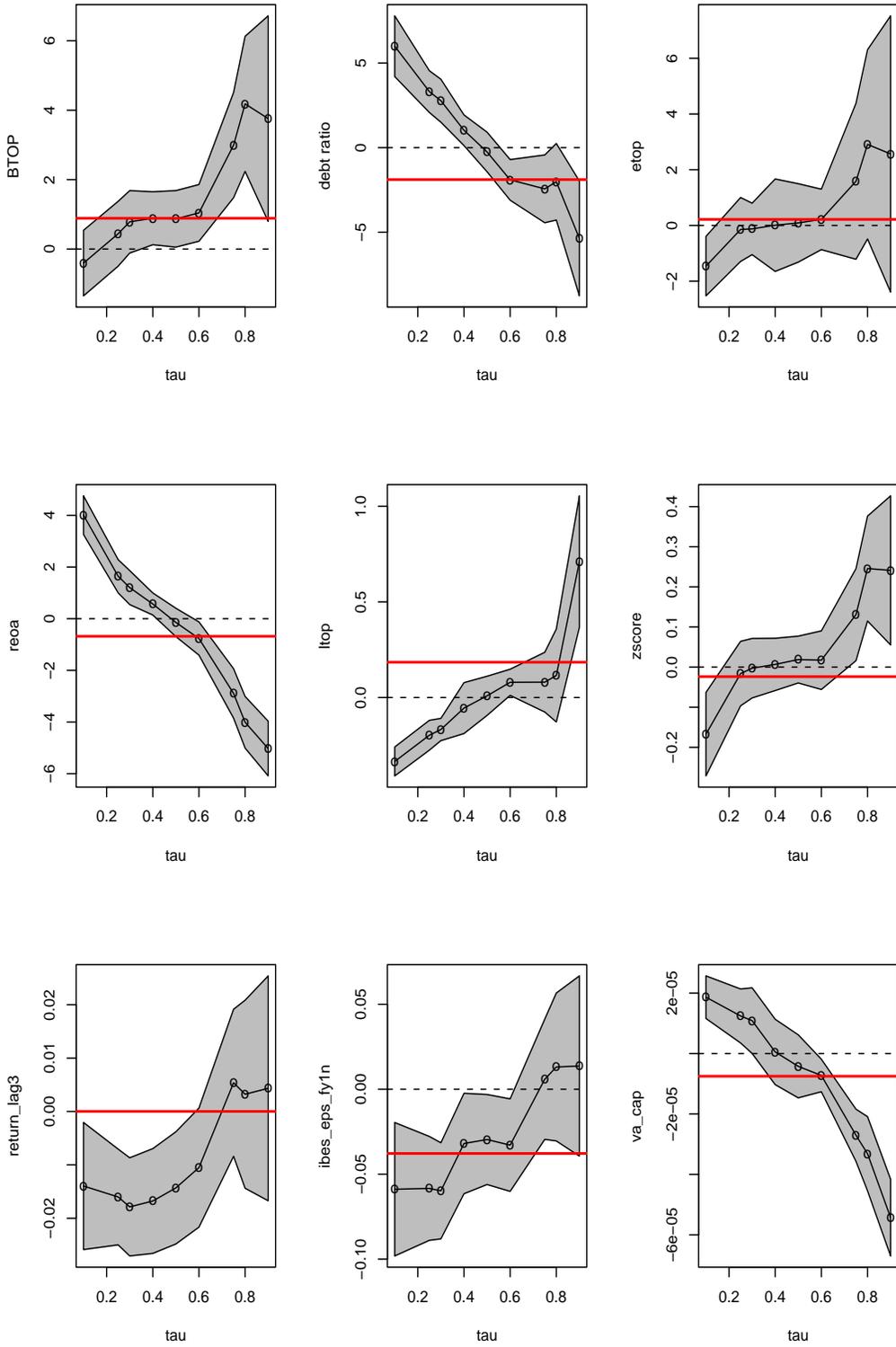


FIGURE 4.1. Quantile regression results at $\tau = \{0.1, 0.25, 0.3, 0.4, 0.5, 0.6, 0.75, 0.8, 0.9\}$. The bold red line is the result from OLS.

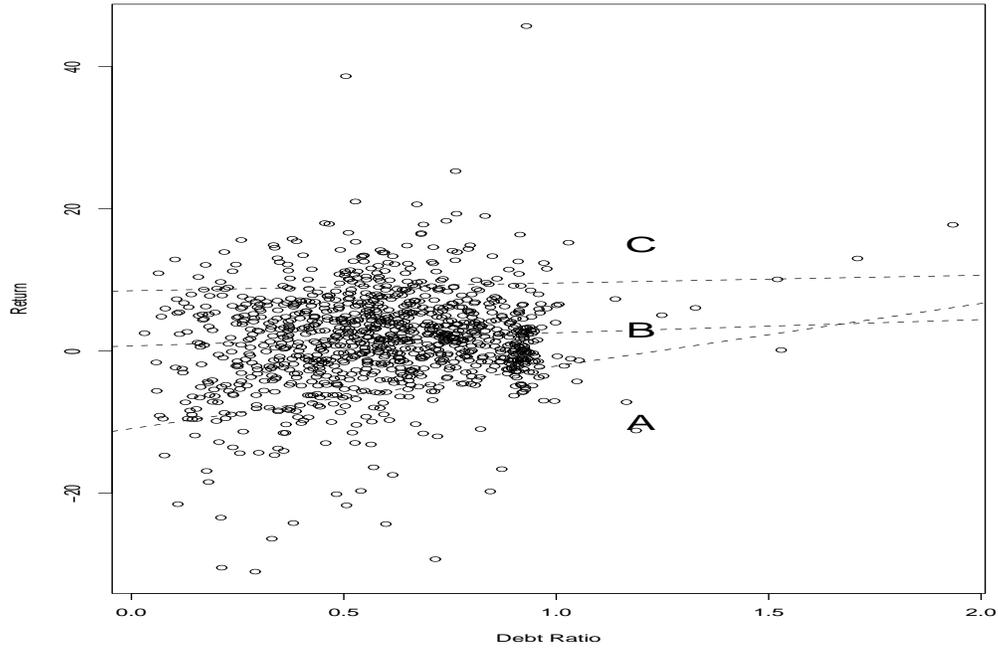


FIGURE 5.1. QRAD-Location. The dashed lines are the quantile regression results at $\tau = \{0.1, 0.5, 0.9\}$. The letter A, B and C indicates the area below the line with $\tau = 0.1$, between the lines with $\tau = 0.1$ and $\tau = 0.9$, and above the line with $\tau = 0.9$, respectively.

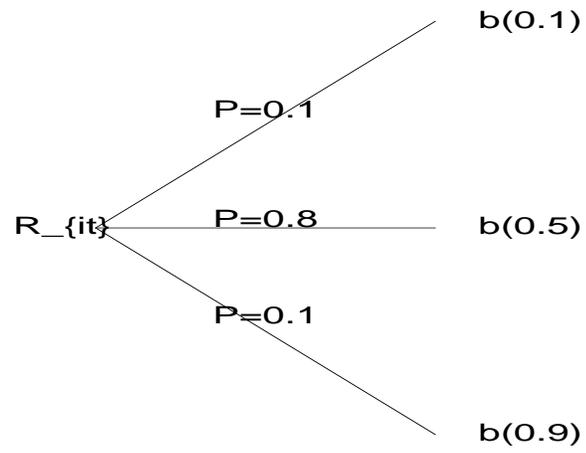


FIGURE 5.2. Probability of QRAD-Location.