Solution 1.7: Take $\Omega = \mathbf{R}$ with its Borel sets, $f_\alpha = \chi_{\{\alpha\}}$, and note that $\sup_{\alpha \in E} f_\alpha = \chi_E$ is not Borel-measurable, if E is not a Borel set (recall Remark 1.3, last sentence).

Solution 1.10: *(i).* If $c = 0$ there is nothing to prove; if $c > 0$ we have

$$\{cf > \alpha\} \equiv \{\omega \in \Omega \,|\, cf(\omega) > \alpha\} = \{\omega \in \Omega \,|\, f(\omega) > \alpha/c\} \in \mathcal{F},$$

and the case $c < 0$ is similar.

(ii). If $\alpha \geq 0$, we have $\{f^2 > \alpha\} = \{f > \sqrt{\alpha}\} \cup \{f < -\sqrt{\alpha}\} \in \mathcal{F}$; if $\alpha < 0$, then $\{f^2 > \alpha\} = \Omega$.

(iii). For every rational number $\varrho \in \mathbf{Q}$ we have $C_\varrho := \{f > \varrho\} \cap \{g > \alpha - \varrho\} \in \mathcal{F}$. Now observe that we have $\{f + g > \alpha\} \equiv \{f > \alpha - g\} = \cup_{\varrho \in \mathbf{Q}} C_\varrho \in \mathcal{F}$.

(iv). Follows from parts (i)-(iii) and $fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$.

(v). For every $\alpha \geq 0$ we have $\{|f| > a\} = \{f > \alpha\} \cup \{f < -\alpha\} \in \mathcal{F}$; if $\alpha < 0$, then $\{|f| > \alpha\} = \Omega$.

(vi). Observe $f^+ = \frac{1}{2}(f + |f|)$, $f^- = \frac{1}{2}(|f| - f)$ and use parts (i), (iii) and (v).

Solution 2.1: $\int |f|^p \, d\mu \geq \int_{\{|f| \geq a\}} |f|^p \, d\mu \geq a^p \cdot \mu(|f| \geq a)$.
Now $\{f \neq 0\} = \cup_{n=1}^\infty \{|f| \geq 1/n\}$ and $\mu(|f| \geq 1/n) \leq n^p \cdot I(|f|^p) < \infty$ if $I(|f|^p) < \infty$, so $\{f \neq 0\}$ is $\sigma-$finite.

Solution 2.2: Without loss of generality, assume $m = 1$ and write $E_1 \setminus F_\infty = \cup_{k=1}^\infty F_k$ for the pairwise-disjoint sets $F_\infty = \cap_{n=1}^\infty E_n$ and $F_k = E_k \setminus E_{k+1}$ ($k \in \mathbf{N}$). Now repeat the argument of (2.8).

Solution 2.3: *(i).* If $f = \sum_{j=1}^m \alpha_j \chi_{E_j}$ is simple, then obviously $I(f) = 0 \Leftrightarrow \alpha_j \mu(E_j) = 0,\ \forall\, j = 1, \cdots, m \Leftrightarrow f = 0,\ \mu-$a.e. For a general $f \in \mathbf{L}^+$ with $\mu(f \neq 0) = 0$, we have also $\varphi = 0,\ \mu-$a.e. for every simple φ with $0 \leq \varphi \leq f$, thus $I(f) = 0$ from (1.3).

If $I(f) = 0$, then $F_n := \{f > 1/n\}$, $n \in \mathbf{N}$ defines a sequence of sets which increase to $F := \{f > 0\}$, with $I(f) \geq I(f\chi_{F_n}) \geq (1/n) \cdot \mu(F_n) \geq 0$ for every $n \in \mathbf{N}$. Thus $\mu(F_n) = 0$, and (2.5) gives $\mu(F) = 0$.

(iv) We have $E = \cap_{n=1}^\infty E_n$, where $E_n := \{f > n\}$, $n \in \mathbf{N}$ defines now a decreasing sequence with $n \cdot \mu(E_n) \leq I(f\chi_{E_n}) \leq I(f) < \infty$. From this and (2.15), we conclude $\mu(E) = \lim_n \mu(E_n) = 0$. On the other hand, we have $F = \cup_{n=1}^\infty F_n$ in the notation of *(i)*, and $\mu(F_n) \leq n\, I(f) < \infty$ for every $n \in \mathbf{N}$.

Solution 2.3: *(vi).* It is clear that ν is a measure for f simple. Otherwise, consider an increasing sequence $\{g_n\} \subseteq \mathcal{S}$ of simple functions with the property (2.12), and note $\nu_n(E) := \int_E g_n \, d\mu \uparrow \int_E f \, d\mu = \nu(E)$, by the Monotone Convergence Theorem. Take disjoint sets $\{G_n\}_{n \in \mathbf{N}} \subseteq \mathcal{F}$, let $G := \cup_{n=1}^\infty G_n$, observe

$$\sum_{j=1}^M \nu_n(G_j) \leq \sum_{j=1}^\infty \nu_n(G_j) = \nu_n(G) \leq \sum_{j=1}^\infty \nu(G_j), \quad \forall\, n \in \mathbf{N}$$

and let $n \to \infty$ to obtain $\sum_{j=1}^M \nu(G_j) \leq \nu(G) \leq \sum_{j=1}^\infty \nu(G_j)$, for all $M \in \mathbf{N}$. Now let $M \to \infty$, and countable additivity follows.

The property $\int g\,d\nu = \int fg\,d\mu$ is obvious, if g is simple. If not, recall

$$\int g\,d\nu \;=\; \sup_{\substack{\varphi\in\mathcal{S}\\ 0\le\varphi\le g}} \int \varphi\,d\nu \;=\; \sup_{\substack{\varphi\in\mathcal{S}\\ 0\le\varphi\le g}} \int \varphi f\,d\mu \;\le\; \sup_{\substack{\psi\in\mathbf{L}^+\\ 0\le\psi\le fg}} \int \psi\,d\mu \;=\; \int fg\,d\mu$$

from *(iii)*; on the other hand, $\int g\,d\nu \ge \int g\,d\nu_n = \int f_n g\,d\mu$ holds for every $n\in\mathbf{N}$ thanks to *(v)* and the fact that f_n is simple, so that $\int g\,d\nu \ge \int fg\,d\mu$ follows, by Monotone Convergence.

Solution 2.4: *(iii).* For the implication $(\Leftarrow)$ in the first equivalence, note

$$\big|I\left(f\chi_E\right) - I\left(g\chi_E\right)\big| \;\le\; \big|I\left((f-g)\cdot\chi_E\right)\big| \;\le\; I(|f-g|)\,, \quad \forall\; E\in\mathcal{F}\,.$$

For the reverse implication $(\Rightarrow)$ in this equivalence, take successively $E=\{f>g\}$, $E=\{f\le g\}$ to obtain $I(|f-g|) = I\left((f-g)\cdot\chi_{\{f>g\}}\right) + I\left((g-f)\cdot\chi_{\{f\le g\}}\right) = 0$, thanks to Exercise 2.3(i).

(iv) Consider the measurable functions $g_m := \sum_{n=1}^m f_n\,,\ h_m := \sum_{n=1}^m |f_n| \uparrow \sum_{n=1}^\infty |f_n| =: h$ with $|g_m|\le h_m\le h$ for all $m\in\mathbf{N}$. From Exercise 2.3(ii),(iv) we have $I(h) = I\left(\sum_{n=1}^\infty |f_n|\right) = \sum_{n=1}^\infty I(|f_n|) < \infty$ and the set $E=\{h=\infty\}$ has $\mu(E)=0$. Thus the function $g(\omega) := \lim_{m\to\infty} g_m(\omega),\ \omega\in E^c$ and $g(\omega):=0,\ \omega\in E$ satisfies $I(g) = \lim_{m\to\infty} I(g_m)$, or equivalently $I\left(\sum_{n=1}^\infty f_n\right) = \lim_{m\to\infty} I\left(\sum_{n=1}^m f_n\right) = \lim_{m\to\infty}\sum_{n=1}^m I\left(f_n\right) = \sum_{n=1}^\infty I\left(f_n\right)$ by Dominated Convergence.

Solution 2.5: For (i), observe $\liminf_n E_n = \cup_{n\ge1}\cap_{k\ge n} E_k = \cup_{n\ge1} F_n$, with $F_n := \cap_{k\ge n} E_k$, $n\ge1$ an increasing sequence. Therefore,

$$\mu(\liminf_n E_n) \;=\; \mu\left(\cup_{n=1}^\infty F_n\right) \;=\; \lim_n \mu(F_n) \;\le\; \liminf_n \mu(E_n)\,,$$

using the continuity from below property (2.5). Similarly for (ii), using the continuity from above property (2.15).

As for (iii), $\sum_{n=1}^\infty \mu(E_n)<\infty$ implies $\mu\left(\cup_{n=1}^\infty E_n\right)<\infty$, and using continuity from above along with subadditivity, one gets:

$$\mu(\limsup_n E_n) \;=\; \lim_n \mu\left(\cup_{k=n}^\infty E_k\right) \;\le\; \lim_n \sum_{k=n}^\infty \mu(E_k) \;=\; 0\,.$$

Solution 2.6: Just apply (2.12) to obtain increasing sequences $\{g_n^{(\pm)}\}$ of simple functions, with $0\le g_1^{(\pm)}\le\ldots g_n^{(\pm)} \longrightarrow f^\pm$ pointwise; then verify that $g_n := g_n^{(+)} - g_n^{(-)}$ have the desired properties.

Solution 1.9: *(ii).* If $g = h\circ f$ for some $h:\mathbf{R}\to\mathbf{R}$, then $g^{-1}(E) = f^{-1}(h^{-1}(E)) = f^{-1}(B)$ for $B := h^{-1}(E)\in\mathcal{B}(\mathbf{R})$, for arbitrary $E\in\mathcal{B}(\mathbf{R})$. In other words, $\{g^{-1}(E)\,;\,E\in\mathcal{B}(\mathbf{R})\}\subseteq\{f^{-1}(B)\,;\,B\in\mathcal{B}(\mathbf{R})\}$, or $\sigma(g)\subseteq\sigma(f)$.

Now start by assuming $\sigma(g)\subseteq\sigma(f)$. Suppose first that g is simple, i.e., $g=\sum_{j=1}^m a_j\,\chi_{E_j}$ with $\{a_j\}_{j=1}^m\subset\mathbf{R}$, and $\{E_j\}_{j=1}^m\subset\mathcal{F}$ disjoint with Ω as their union. We have that $E_j\in\sigma(g)\subseteq\sigma(f) = f^{-1}(\mathcal{B}(\mathbf{R}))$ is then of the form $E_j = f^{-1}(B_j)$ for some $B_j\in\mathcal{B}(\mathbf{R}))$, $j=1,\cdots,m$, thus $g(\omega) = \sum_{j=1}^m a_j\,\chi_{f^{-1}(B_j)}(\omega) = \sum_{j=1}^m a_j\,\chi_{B_j}(f(\omega)) = h(f(\omega))$, where the simple function $h:=\sum_{j=1}^m a_j\,\chi_{B_j}$ is $\mathcal{B}(\mathbf{R})-$measurable. For Borel-measurable $g:\mathbf{R}\to[0,\infty)$, take a sequence of simple, $\sigma(f)-$measurable functions $\{g_n\}_{n\in\mathbf{N}}$ increasing to g pointwise, with $g_n = h_n\circ f$ for some simple, Borel-measurable

$h_n : \mathbf{R} \to [0, \infty)$; now let $h := \limsup_n h_n$ and observe that $g = \lim_n g_n = \lim_n (h_n \circ f) = h \circ f$. Finally, decompose an arbitrary Borel-measurable $g : \mathbf{R} \to \mathbf{R}$ as $g = g^+ - g^-$, and repeat the above procedure to each of $g^\pm$.

Solution 3.1: For any $D \in \mathcal{G}$, $E \in \mathcal{G}$, we have $D \cup E \in \mathcal{E}$. Indeed, the complement of E can be written as a finite union $E^c = \cup_{j=1}^n F_j$ of pairwise-disjoint sets $\{F_j\}_{j=1}^n \subseteq \mathcal{G}$; thus $D \setminus E = \cup_{j=1}^n (D \cap F_j)$, and $D \cup E = (D \setminus E) \cup E = E \cup \big(\cup_{j=1}^n (D \cap F_j) \big)$ is a finite union of disjoint sets in $\mathcal{G}$. By induction, it is seen that for any $\{E_k\}_{k=1}^m \subseteq \mathcal{G}$, the union $\cup_{k=1}^m E_k$ can be written as a finite disjoint union of sets from $\mathcal{G}$, and thus belongs to $\mathcal{E}$. It follows that $\mathcal{E}$ is closed under finite unions. To see that $\mathcal{E}$ is also closed under complementation, take $\{E_k\}_{k=1}^m \subseteq \mathcal{G}$ with $E_k^c = \cup_{j=1}^n F_k^{(j)}$ a finite union of disjoint subsets from $\mathcal{G}$, for each $k = 1, \cdots, m$; then $(\cup_{k=1}^m E_{k=1}^m)^c = \cap_{k=1}^m \left(\cup_{j=1}^n F_k^{(j)} \right) = \cup \{ F_1^{(j_1)} \cup \cdots \cup F_m^{(j_m)} ; j_1, \cdots, j_m = 1, \cdots, n \}$ is a disjoint union of sets from $\mathcal{G}$, therefore belongs to $\mathcal{E}$.

Solution 3.4: *(i).* The class $\mathcal{G}$ of null sets is closed under countable unions, and thus so is $\bar{\mathcal{F}}$; indeed, if $\{E_n\} \subseteq \mathcal{F}$, $\{A_n\} \subseteq \mathcal{G}$ and $F_n \subseteq A_n$ for every $n \in \mathbf{N}$, then $\cup_n (E_n \cup F_n) = E \cup F$, where $E := \cup_n E_n \in \mathcal{F}$ and $F := \cup_n F_n \subseteq \cup_n A_n \in \mathcal{G}$.

Now $\overline{\mathcal{F}}$ is also closed under complementation; to see this, take $E \cup F \in \overline{\mathcal{F}}$ with $E \in \mathcal{F}$, $F \subseteq A \in \mathcal{G}$, assume $E \cap A = \emptyset$ (otherwise, replace F, A by $F \setminus E$, $A \setminus E$) and write $(E \cup F)^c = (E \cup A)^c \cup (A \setminus F) \in \overline{\mathcal{F}}$, because $A \setminus F \in \mathcal{N}$. Thus $\overline{\mathcal{F}}$ is a σ-algebra.

If $\overline{E} = E_i \cup F_i$ with $E_i \in \mathcal{F}_i$, $F_i \subseteq A_i \in \mathcal{G}$ $(i = 1, 2)$, then $\mu(E_1) \le \mu(E_2) + \mu(A_2) = \mu(E_2)$; similarly, $\mu(E_2) \le \mu(E_1)$, thus $\bar{\mu}$ is well-defined on $\bar{\mathcal{F}}$. It is checked easily that $\bar{\mu}$ agrees with μ on $\mathcal{F}$. To verify that $\bar{\mu}$ is countably additive, take a sequence $\{E_n \cup F_n\}_{n \in \mathbf{N}}$ of pairwise-disjoint sets with $E_n \in \mathcal{F}$, $F_n \subseteq A_n \in \mathcal{G}$, $E_n \cap A_n = \emptyset$ and observe $\bar{\mu}\left(\cup_{n=1}^\infty (E_n \cup F_n) \right) = \mu\left(\cup_{n=1}^\infty E_n \right) = \sum_{n=1}^\infty \mu(E_n) = \sum_{n=1}^\infty \bar{\mu}(E_n \cup F_n)$. Clearly, $\mathcal{N} \subseteq \overline{\mathcal{F}}$, so $\bar{\mu}$ is a complete measure on $\overline{\mathcal{F}}$.

Suppose ν is another measure on $\overline{\mathcal{F}}$ that agrees with μ on $\mathcal{F}$. To prove $\nu = \bar{\mu}$, consider arbitrary $E \in \mathcal{F}$, $F \subseteq A \in \mathcal{G}$ and observe

$$\mu(E) = \nu(E) \le \nu(E \cup F) \le \nu(E \cup A) = \mu(E \cup A) \le \mu(E) + \mu(A) = \mu(E),$$

thus $\nu(E \cup F) = \mu(E)$ and $\nu \equiv \bar{\mu}$.

Solution 3.5: *(i).* If $\mu^*(E) = 0$, we have by monotonicity $\mu^*(A \cap E) = 0$ as well, for every $A \subseteq \Omega$, and thus $\mu^*(A) \ge \mu^*(A \cap E^c) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$. In other words $E \in \mathcal{M}$, and the restriction of μ^* to $\mathcal{M}$ is a complete measure.

Solution 3.6: *(i).* For any given $B \in \mathcal{B}(\mathbf{R})$ we have by assumption $A := \{g \in B, \ f \ne g\} \subseteq \{f \ne g\} \in \mathcal{N} \subseteq \mathcal{F}$, since the space is complete. Thus $\{f = g\} \in \mathcal{F}$, and $\{g \in B\} = \{f \in B, \ f = g\} \cup A \in \mathcal{F}$, since f is measurable.

Solution 3.7: Clearly $\mathcal{E} \subseteq \mathcal{M} := m(\mathcal{E}) \subseteq \sigma(\mathcal{E})$; and in order to show the reverse inclusion $\sigma(\mathcal{E}) \subseteq \mathcal{M}$, it suffices to prove that $\mathcal{M}$ is a $\sigma-$algebra. Indeed, as we shall see below, for any $F \in \mathcal{M}$, $G \in \mathcal{M}$ the sets

$$F \setminus G, \quad G \setminus F, \quad F \cap G \qquad \text{belong to } \mathcal{M}, \tag{10.1}$$

and because $\Omega \in \mathcal{E}$ we deduce that $\mathcal{M}$ is an algebra. Now, for any $\{E_n\}_{n \in \mathbf{N}}$, the sets $F_n := \cup_{j=1}^n E_j$, $n \in \mathbf{N}$ belong to $\mathcal{M}$, and $\cup_{j \in \mathbf{N}} E_j = \cup_{n \in \mathbf{N}} F_n = \lim_n \uparrow, F_n \in \mathcal{M}$, so $\mathcal{M}$ is indeed a

$\sigma-$algebra. To see the property $(*)$, fix an arbitrary $G \in \mathcal{M}$ and consider the class $\mathcal{C}(G) := \{F \in \mathcal{M} \,|\, (10.1) \text{ holds}\}$. This contains $\emptyset$ and G, is a monotone class, and $F \in \mathcal{C}(G)$ implies $G \in \mathcal{C}(F)$. Also, for $G \in \mathcal{E}$, we have $F \in \mathcal{E} \Rightarrow F \in \mathcal{C}(G)$ (because $\mathcal{E}$ is an algebra), thus $\mathcal{E} \subseteq \mathcal{C}(G)$ and $\mathcal{M} \subseteq \mathcal{C}(G)$; in other words, $\mathcal{E} \subseteq \mathcal{M} \subseteq \cap_{G \in \mathcal{E}} \mathcal{C}(G)$. In other words, for every $G \in \mathcal{M}$ we have: $G \in \mathcal{C}(F)$, $\forall\, F \in \mathcal{E}$, which implies $F \in \mathcal{C}(G)$, $\forall\, F \in \mathcal{E}$, which implies $\mathcal{E} \subseteq \mathcal{C}(G)$, which implies $\mathcal{M} \subseteq \mathcal{C}(G)$ because $\mathcal{C}(G)$ is a monotone class. We conclude that $\mathcal{M} \equiv \mathcal{C}(G)$.

Solution 3.8: *(i).* If $\mathcal{D}$ is both a $\pi-$system and a $\lambda-$system, then it is closed under complementation and finite unions. Indeed, for every sequence $\{E_n\}_{n \in \mathbf{N}} \subseteq \mathcal{D}$ we have $E_i^c = \Omega \backslash E_1 \in \mathcal{D}$ and $E_1 \cup E_2 \big(E_1^c \cap E_2^c\big)^c \in \mathcal{D}$. To show that $\mathcal{D}$ is closed also under countable unions, just observe that $G_n := \cup_{j=1}^n E_j \in \mathcal{D}$ for every integer n and $G_n \uparrow \cup_{j=1}^\infty E_j =: G$, so that $G \in \mathcal{D}$ as well. The reverse implication is trivial.

The intersection of an arbitrary collection of $\lambda-$systems is also a $\lambda-$system; so for any collection $\mathcal{A} \subseteq \mathcal{F}$ of subsets of Ω we can define $\lambda(\mathcal{A})$ as the intersection of all $\lambda-$systems that contain $\mathcal{A}$. This is the smallest $\lambda-$system that contains $\mathcal{A}$, and clearly $\mathcal{A} \subseteq \lambda(\mathcal{A}) \subseteq \sigma(\mathcal{A})$.

(ii). Now let us show $\lambda(\mathcal{D}) = \sigma(\mathcal{D})$ for any $\pi-$system $\mathcal{D}$. In particular, that *any $\lambda-$system which contains a $\pi-$system also contains the $\sigma-$algebra generated by it.*

Thanks to the above discussion we need only show that $\mathcal{A} := \sigma(\mathcal{D})$ is a $\pi-$system; that is, closed under pairwise intersections. Consider first the class

$$\mathcal{A}_1 := \{ A \in \mathcal{A} \,|\, A \cap B \in \mathcal{A} ,\ \forall\, B \in \mathcal{D} \}.$$

Because $\mathcal{D}$ is a $\pi-$system we have $\mathcal{D} \subseteq \mathcal{A}_1$. We also can check that $\mathcal{A}_1$ is a $\lambda-$system, because so is $\mathcal{A}$. Since $\mathcal{A}$ is the smallest $\lambda-$system that contains $\mathcal{D}$, this shows that $\mathcal{A}_1 = \mathcal{A}$. Next, let us look at the class

$$\mathcal{A}_2 := \{ A \in \mathcal{A} \,|\, A \cap B \in \mathcal{A} ,\ \forall\, B \in \mathcal{A} \}$$

and deduce $\mathcal{D} \subseteq \mathcal{A}_2$ from $\mathcal{A}_1 = \mathcal{A}$. We also check that $\mathcal{A}_2$ is a $\lambda-$system, so $\mathcal{A}_2 = \mathcal{A}$ and thus $\mathcal{A}$ is a $\pi-$system.

(iii). The class $\mathcal{E} := \{ E \in \mathcal{F} \,|\, \mu(E) = \nu(E) \}$ is a $\lambda-$system. Indeed, $\Omega \in \mathcal{E}$ by assumption; and if A, B with $B \subseteq A$ are in $\mathcal{E}$, we have $\mu(A \setminus B) = \mu(A) - \mu(B) = \nu(A) - \nu(B) = \nu(A \setminus B)$ because μ, ν are finite measures (the finiteness assumption is crucial here), so $A \setminus B \in \mathcal{E}$; whereas for any increasing sequence $\{E_n\} \subseteq \mathcal{E}$ with $E := \cup_{n=1}^\infty E_n \in \mathcal{F}$ we have $\mu(E) = \lim_n \uparrow \mu(E_n) = \lim_n \uparrow \nu(E_n) = \nu(E)$ from (2.5), so $E \in \mathcal{E}$. By assumption $\mathcal{D} \subseteq \mathcal{E}$, and from part *(i)* we get $\sigma(\mathcal{D}) = \lambda(\mathcal{D}) \subseteq \mathcal{E}$, Q.E.D.

Solution 4.1: For the first claim, denote its right-had side by $\rho(E)$. If $E \subseteq \bigcup_{n \in \mathbf{N}} (a_n, b_n)$, let $\lambda_n := b_n - a_n$, $I_n^{(k)} := (b_n - \lambda_n 2^{1-k}, b_n - \lambda_n 2^{-k})$ for $k \in \mathbf{N}$, so that

$$(a_n, b_n) = \bigcup_{n \in \mathbf{N}} I_n^{(k)}, \quad E \subseteq \bigcup_{n \in \mathbf{N}} \bigcup_{k \in \mathbf{N}} I_n^{(k)} \quad \text{and} \quad \sum_{n \in \mathbf{N}} \overline{\mu}_F((a_n, b_n)) = \sum_{n \in \mathbf{N}} \sum_{k \in \mathbf{N}} \overline{\mu}_F(I_n^{(k)}) \geq \overline{\mu}_F(E).$$

It follows that $\rho(E) \geq \overline{\mu}_F(E)$. For the reverse inequality, given any $\delta > 0$ we find a sequence $\{(a_n, b_n)\}_{n \in \mathbf{N}}$ with $E \subseteq \bigcup_{n \in \mathbf{N}} (a_n, b_n)$ and $\sum_{n \in \mathbf{N}} \overline{\mu}_F((a_n, b_n)) \leq \overline{\mu}_F(E) + \delta$ from $(4.1)'$, and for each $n \in \mathbf{N}$ a $\zeta_n > 0$ such that $F(b_n + \zeta_n) - F(b_n) < \delta 2^{-n}$. Then we have $E \subseteq \bigcup_{n \in \mathbf{N}} (a_n, b_n + \zeta_n)$ and

$$\sum_{n \in \mathbf{N}} \overline{\mu}_F((a_n, b_n + \zeta_n)) \leq \sum_{n \in \mathbf{N}} \left[\overline{\mu}_F((a_n, b_n]) + \delta 2^{-n} \right] \leq \sum_{n \in \mathbf{N}} \overline{\mu}_F((a_n, b_n]) + \delta \leq \overline{\mu}_F(E) + 2\delta,$$

and $\rho(E) \le \overline{\mu}_F(E)$ follows.

. If U is open and $E \subseteq U$, clearly $\overline{\mu}_F(E) \le \overline{\mu}_F(U)$ and $\overline{\mu}_F(E) \le \inf_{\substack{U \in \mathcal{O} \\ U \supseteq E}} \overline{\mu}_F(U)$. The reverse inequality follows from (4.1)$'$, once we consider every $U \in \mathcal{O}$ with $U \supseteq E$ as a countable union of open intervals, so that $\overline{\mu}_F(U) \le \sum_{n \in \mathbf{N}} \overline{\mu}_F((a_n, b_n))$.

. For the third claim, suppose first that E is bounded. If it is also closed (i.e., $\overline{E} = E$), then E is compact and there is nothing to prove. If not, given any $\delta > 0$ we can choose $U \in \mathcal{O}$, $U \supseteq \overline{E} \setminus E$ with $\overline{\mu}_F(U) \le \overline{\mu}_F(\overline{E} \setminus E) + \delta$. Then $K := \overline{E} \setminus U$ is compact, $K \subseteq E$ and

$$\overline{\mu}_F(K) = \overline{\mu}_F(E) - \overline{\mu}_F(E \cap U) = \overline{\mu}_F(E) - \big(\overline{\mu}_F(U) - \overline{\mu}_F(U \setminus E)\big) \ge \overline{\mu}_F(E) - \overline{\mu}_F(U) + \overline{\mu}_F(\overline{E} \setminus E)\big) \ge \overline{\mu}_F(E) - \delta . \ \blacksquare$$

If E is unbounded, consider $E_n := E \cap (n, n+1)$. From what has been shown, for every $\delta > 0$ and $n \in \mathbf{N}$ there exists a compact set $K_n \subseteq E_n$ with $\overline{\mu}_F(K_n) \ge \overline{\mu}_F(E_n) - \delta 2^{-n}$. The set $C_n := \bigcup_{j=-n}^{n} K_j$ is compact, it is contained in E, and we have

$$\overline{\mu}_F(C_n) \ge \overline{\mu}_F\Big(\bigcup_{j=-n}^{n} E_j \Big) - \varepsilon . \quad \text{But} \quad \overline{\mu}_F(E) = \lim_n \overline{\mu}_F\Big(\bigcup_{j=-n}^{n} E_j \Big), \quad \text{and the result follows.}$$

Solutions 4.5, 4.6: *(ii)* For a given partition Π, the simple functions $\bar{g}^{\Pi} := \sum_{j=1}^{n} \overline{M}_j \, \chi_{(t_{j-1}, t_j]}$ and $\underline{g}^{\Pi} := \sum_{j=1}^{n} \underline{M}_j \, \chi_{(t_{j-1}, t_j]}$ satisfy $\underline{g}^{\Pi} \le f \le \bar{g}^{\Pi}$ as well as $\underline{S}(f;\Pi) \equiv I(\underline{g}^{\Pi}) \le I(\bar{g}^{\Pi}) \equiv \overline{S}(f;\Pi)$. If f is Riemann-integrable, there exists a *nested* sequence of partitions $\big\{\Pi^{(k)}\big\}_{k \in \mathbf{N}}$, with mesh $||\Pi^{(k)}|| := \max_{1 \le j \le n^{(k)}} (t_j^{(k)} - t_{j-1}^{(k)}) \longrightarrow 0$ as $k \to \infty$, such that

$$\lim_{k \to \infty} I(\underline{g}^{(k)}) = \lim_{k \to \infty} I(\bar{g}^{(k)}) = R(f) , \qquad \text{where } \ \underline{g}^{(k)} \equiv \underline{g}^{\Pi^{(k)}} , \ \ \bar{g}^{(k)} \equiv \bar{g}^{\Pi^{(k)}} .$$

Now the limits $\underline{g} := \lim_{k \to \infty} \uparrow \underline{g}^{(k)} \le f \le \lim_{k \to \infty} \downarrow \bar{g}^{(k)} =: \bar{g}$ exist and are Lebesgue-measurable, as limits of monotone sequences of simple functions. Thus $I(\underline{g}) = \lim_{k \to \infty} \uparrow I(\underline{g}^{(k)}) \le \lim_{k \to \infty} \downarrow I(\bar{g}^{(k)}) = I(\bar{g})$, by the Dominated Convergence Theorem. It follows that $I(\underline{g}) = I(\bar{g}) = R(f)$, thus $\underline{g} = \bar{g} \ (= f)$, $\bar{\lambda}$–a.e. Since $\underline{g}$ $(\bar{g})$ are Lebesgue-measurable *and* $(\mathbf{R}, \mathcal{L}, \bar{\lambda})$ *is complete*, it follows from Exercise 3.6 that f is Lebesgue-measurable as well. But then f is Lebesgue-integrable, and $I(f) = I(\underline{g}) = I(\bar{g}) = R(f)$.

For the function $f = \chi_{\mathbf{Q}}$, the Darboux sums are $\overline{S}(f;\Pi) \equiv 1$ and $\underline{S}(f;\Pi) \equiv 0$ across partitions, so $\underline{R}(f) = 0$, $\bar{R}(f) = 1$ so the Riemann integral does not exist. On the other hand, $\mathbf{Q}$ is clearly a Borel set (countable union of singletons), so the simple function $f = \chi_{\mathbf{Q}}$ is Borel-measurable and $I(f) = \lambda(\mathbf{Q}) = \sum_{q \in \mathbf{Q}} \lambda(\{q\}) = 0$.

Solution 4.10: There exists a compact set $K \subset I$ with $t < \lambda(K)$; recall the regularity property (1.9). Then $K \subset \cup_{i=1}^{n} J_i$ for some $\{J_1, \cdots, J_n\} \subset \mathcal{U}$ enumerated so that $\lambda(J_1) \ge \lambda(J_2) \ge \cdots \ge \lambda(J_n)$. Put $I_1 := J_1$; for $j = 2, 3, \cdots$, select $I_j := J_{m(j)}$, where $m = m(j)$ is the smallest index for which J_m does not intersect any $I_1, \cdots, I_{j-1}$.

Let L_j be the interval with the same center as I_j but three times as long. Then either each J_i is one of $I_1, \cdots, I_n$; or else J_i intersects $I_j = J_\ell$ for some $\ell < k$, so that $\lambda(J_i) \le \lambda(J_\ell)$ and $J_i \subseteq L_j$. Then, with r the largest index j for which I_j is defined:

$$t < \lambda(K) < \lambda\left(\cup_{i=1}^{n} J_i\right) \le \sum_{j=1}^{r} \lambda(L_j) = 3 \cdot \sum_{j=1}^{r} \lambda(I_j) .$$

Solution 4.11: It suffices to show that $\lim_{\delta\downarrow 0} \frac{1}{\delta}\, \mu\big((x-\delta, x+\delta)\big) = 0$ holds for $\lambda-$a.e. $x \in A$. Define

$$F_k := \left\{ x \in A \,\middle|\, \overline{\lim}_{\substack{\delta\downarrow 0 \\ \delta\in\mathbf{Q}}} \frac{\mu\big((x-\delta, x+\delta)\big)}{\delta} > \frac{1}{k} \right\},$$

a measurable set for each $k \in \mathbf{N}$ (justify!). For each $\varepsilon > 0$, there exists an open set V with $A \subset V$ and $\mu(V) < \varepsilon$ (the regularity property of (1.9)). For every $x \in F_k$, there exists a rational number $\delta > 0$ such that $(x-\delta, x+\delta) \subset V$ and $\mu\big((x-\delta, x+\delta)\big) > \delta/k$. Such intervals $(x-\delta, x+\delta)$ cover F_k; thus, from Exercise 4.7, for *any* given $t < \lambda(F_k)$ there exist finitely many disjoint subintervals $I_1, \cdots, I_r$ of V with

$$t \le 3 \cdot \sum_{j=1}^{r} \lambda(I_j) \le 6k \sum_{j=1}^{r} \mu(I_j) \le 6k \cdot \mu(V) \le 6k\,\varepsilon.$$

In other words, $\lambda(F_k) \le 6k\,\varepsilon$, so $\lambda(F_k) = 0$ for every $k \in \mathbf{N}$ (just let $\varepsilon \downarrow 0$), and then let $k \to \infty$ to conclude.

Solution 4.14: The second equation clearly follows from the first. If $\mu = \mu^+ - \mu^-$ is the signed measure associated with the function A, and $\nu = \nu^+ - \nu^-$ the signed measure associated with the function B, then both sides of the first equation express the product measure $(\mu \otimes \nu)\big([0,t]^2\big)$.

Indeed, this is very clear for the left-hand side. As for the right-hand side, $\int_0^t A(s)\, dB(s)$ is the measure of the upper triangle including the diagonal, whereas $\int_0^t B(s-)\, dA(s)$ the measure of the lower triangle excluding the diagonal.

Solution 4.15: The result is true for $\Phi(x) = x$ and, if it is true for some Φ, it is also true for $x \mapsto x\Phi(x)$ by the integration-by parts formula. Thus, the formula is true for polynomials. Now approximate any continuous and continuously differentiable function by polynomials, to get the result.

Solution 4.16: An application of the integration by parts formula to the product of the functions $t \mapsto \prod_{0\le s\le t} \big(1 + \Delta A(s)\big)$ and $t \mapsto e^{\,A^c(t)}$, both right-continuous and of finite variation, shows rather easily that this product is indeed as solution of the integral equation.

Now suppose that $Z(\cdot)$, $\widetilde{Z}(\cdot)$ are solutions of the integral equation; their difference $D(\cdot) := Z(\cdot) - \widetilde{Z}(\cdot)$ solves the equation $D(t) = \int_0^t D(s-)\, dA(s)$, $0 \le t < \infty$. With $V(t)$ denoting the total variation of $A(\cdot)$ on the interval $[0,t]$, and $M(t) := \sup_{0\le s\le t} |D(s)|$, we have then $|D(t)| \le M(t)V(t)$, therefore also

$$|D(t)| \le M(t) \int_0^t V(s-)\, dV(s) \le M(t) \cdot \frac{1}{2}\, V^2(t), \qquad 0 \le t < \infty$$

thanks to the integral equation for $D(\cdot)$ and the integration by parts formula of Exercise 4.14. Iterating this procedure, we obtain

$$|D(t)| \le \frac{M(t)}{n!} \int_0^t V^n(s-)\, dV(s) \le M(t) \cdot \frac{1}{(n+1)!}\, V^{n+1}(t), \qquad 0 \le t < \infty$$

for every integer n, and deduce $D(\cdot) \equiv 0$.

Solution 4.17: The increase of $\Gamma(\cdot)$ and the inequalities $A\big(\Gamma(u)\big) \ge u$, $\Gamma\big(A(t)\big) \ge t$ are quite clear. On the other hand, the set $\{\, t \ge 0 \,|\, A(t) > u\}$ is the union of the sets $\{\, t \ge 0 \,|\, A(t) > u + \varepsilon\}$ over $\varepsilon > 0$, and the right-continuity of $\Gamma(\cdot)$ follows.

Now for $\Gamma(u) > t$ we have $A(t) \le u$; therefore, $A(t) \le \inf\{\, u \ge 0 \,|\, \Gamma(u) > t \,\}$. To obtain an inequality in the reverse direction, observe that we have $\Gamma\big(A(t+\delta)\big) \ge t+\delta > t$, thus also $A(t+\delta) \ge \inf\{\, u \ge 0 \,|\, \Gamma(u) > t \,\}$, for every $\delta > 0$. Now recall that $A(\cdot)$ is right-continuous, to deduce $A(t) \ge \inf\{\, u \ge 0 \,|\, \Gamma(u) > t \,\}$.

For the choice $h(s) = \chi_{[0,t]}(s)$, the change-of-variable formula reads $A(t) = \int_0^\infty \chi_{\{\Gamma(u) \le t\}}\, du$; but this is a consequence of the definition of $\Gamma(\cdot)$. By taking differences, the formula is seen to hold also for indicators of the type $\chi_{(r,t]}$; and by monotone class arguments, for any h with compact support. Taking increasing limits gives the validity of the change of variable formula in the generality claimed.

Solution 5.3: Note that $\{\omega \in \Omega : |f(\omega)| > a\} = \cup_{n=1}^\infty \{\omega \in \Omega : |f(\omega)| > a + (1/n)\}$, and if the sets on the right are all null, then so is the set on the left.

Solution 5.4: From the notion of convergence for sequences of real numbers, we have

$$\{\lim_n g_n = g\} = \cap_{m=1}^\infty \cup_{k=1}^\infty \cap_{n=k}^\infty \{|g_n - g| \le 1/m\} = \cap_{m=1}^\infty C(m)\,,$$

where $C(m) := \cup_{k=1}^\infty B_k(m)$, $B_k(m) := \cap_{n=k}^\infty \{|g_n - g| \le 1/m\}$. Now observe that $\lim_n g_n = g$, $\mu-$a.e. $\Leftrightarrow$ $\mu\left((C(m))^c\right) = 0$, $\forall\, m \in \mathbf{N}$ $\Leftrightarrow$ $\mu\left(\cap_{k=1}^\infty (B_k(m))^c\right) = 0$, $\forall\, m \in \mathbf{N}$, which in turn is equivalent to $\lim_{k\to\infty} \mu\left(\cup_{n=k}^\infty \{|g_n - g| > \frac{1}{m}\}\right) = 0$, $\forall\, m \in \mathbf{N}$. This is because the sequence of sets $\{(B_k(m))^c\}_{k\in\mathbf{N}}$ is decreasing, and the measure μ is finite; recall Exercise 2.2. (If the measure space is not finite but $|g_n| \le f$ holds for all $n \in \mathbf{N}$ for some $f \in \mathbf{L}^1(\mu)$, then $\mu(|g_n - g| > 1/m) \le m\, I(|g_n - g|) \le 2m\, I(f) < \infty$ hold for every $n \in \mathbf{N}$, and the same argument applies.)

Thus, for any given $\delta > 0$, $m \in \mathbf{N}$, there exists $N_m \in \mathbf{N}$ such that $\mu\left((B_k(m))^c\right) < \delta\, 2^{-m}$, $\forall\, k \ge N_m$; set

$$E := \cap_{m=1}^\infty B_{N_m}(m) = \cap_{m=1}^\infty \cap_{k=N_m}^\infty \{|g_k - g| \le 1/m\}\,,$$

and observe that $\mu(E^c) < \delta$, that $\sup_{k \ge N_m} |g_k(\omega) - g(\omega)| \le (1/m)$ holds for every $\omega \in E$, as well as that $\mu\left(|g_k - g| > 1/m\right) \le \delta$ holds for every $k \ge N_m$.

Solution 5.5 : *(i).* If we have $\mu(|g_n - g| > \varepsilon) \to 0$, $\mu(|g_n - h| > \varepsilon) \to 0$ as $n \to \infty$ for every $\varepsilon > 0$, where g and h are measurable functions, then

$$\mu(|g - h| > 2\varepsilon) \;\le\; \mu(|g_n - g| > \varepsilon) + \mu(|g_n - h| > \varepsilon) \;\longrightarrow\; 0\,, \quad \text{as } n \to \infty\,.$$

Thus, $\mu(g \ne h) = \mu(|g - h| > 0) \;=\; \lim_{m\to\infty} \mu(|g - h| > 1/m) \;=\; 0$.

Now suppose that $\mu(\Omega) < \infty$ and $f_n \to f$ $\mu-$a.e.; then the indicator function $g_n := \chi_{\{|f_n - f| > \varepsilon\}}$ is dominated by the integrable function $g \equiv 1$, and $g_n \to 0$ $\mu-$a.e.. Therefore, $\mu\big(|f_n - f| > \varepsilon\big) = I(g_n) \to 0$ by the Dominated Convergence Theorem.

(ii). The function $F(x) = x/(1+x)$, $x \ge 0$ is strictly increasing and concave with $0 \le F(x) \le \min(1, x)$, $F(x+y) \le F(x) + F(y)$. Thus the quantity

$$\rho(g_n, g) \;=\; \int_{\{|g_n - g| > \varepsilon\}} F(|g_n - g|)\, d\mu \;+\; \int_{\{|g_n - g| \le \varepsilon\}} F(|g_n - g|)\, d\mu$$

dominates $\int_{\{|g_n - g| > \varepsilon\}} F(|g_n - g|) d\mu \ge F(\varepsilon) \cdot \mu(\{|g_n - g| > \varepsilon\})$ and is dominated by $(\varepsilon/(1+\varepsilon))\, \mu(\Omega) + \mu(\{|g_n - g| > \varepsilon\})$; this shows the stated equivalence. On the other hand, $\rho(f, g) = 0$ iff $f = g$, $\mu-$a.e., and $\rho(f, g) + \rho(g, h) = I\left(F(|f - g|) + F(|g - h|)\right) \ge I\left(F(|f - g| + |g - h|)\right) \ge I\left(F(|f - h|\right) = \rho(f, h)$.

(iii). From the Čebyšev inequality, we have for every $\varepsilon > 0$: $\mu(|f_n - f| > \varepsilon) \le \varepsilon^{-p} \cdot I(|f_n - f|^p) \longrightarrow 0$, as $n \to \infty$.

(iv). The sequence $g_n = n\,\chi_{(0,1/n]}$ converges to $g \equiv 0$ a.e. with respect to Lebesgue measure λ on $(0,1]$, but $I(g_n) = n\,\lambda(0,1/n) = 1$, $\forall\, n \in \mathbf{N}$: *a.e. convergence does not imply convergence in* $\mathbf{L}^1$.

To see that **a.e.-convergence does not imply convergence in measure, if the space has infinite measure** $\mu(\Omega) = \infty$, take $\Omega = [0,\infty)$ with Lebesgue measure λ, and $f_n(\omega) := \chi_{(n,n+1)}(\omega) \longrightarrow f(\omega) \equiv 0$, $\forall \omega \in \Omega$, as $n \to \infty$. But $\lambda(|f_n - f| > \varepsilon) = \lambda(n, n+1) = 1$, for all $n \in \mathbf{N}$ and $\varepsilon > 0$, so convergence in measure fails.

On the other hand, for $k \in \mathbf{N}$, $j = 0, 1, \ldots, 2^k - 1$, define

$$g_n(\omega) \equiv g_{2^k+j}(\omega) := \chi_{(j2^{-k},(j+1)2^{-k}]}(\omega)\,,\ \omega \in \Omega = (0,1]\,, \qquad \text{with } \ n = 2^k + j\,.$$

For instance, with $k = 2$, we have $f_4 = \chi_{(0,1/4]}$, $f_5 = \chi_{(1/4,1/2]}$, $f_6 = \chi_{(1/2,3/4]}$ and $f_7 = \chi_{(3/4,1]}$, corresponding to $j = 0, \ldots, 3$, respectively. Clearly, $I(f_n) = I(f_{2^k+j}) = 2^{-k}$ for $j = 0, \ldots, 2^k - 1$; thus $\lim_{n\to\infty} I(f_n) = 0$, so that $\{f_n\}_{n\in\mathbf{N}}$ converges to zero, both in $\mathbf{L}^1$ and in measure (thanks to (iii)). However, for any given $\omega \in (0,1)$, we have $f_n(\omega) = 0$ for infinitely many n, as well as $f_n(\omega) = 1$ for infinitely many n, so that $\mu(f_n \to 0) = 0$: *you can have convergence both in measure and in* $\mathbf{L}^1$, *but not a.e.*

(v). Choose a subsequence $\{h_k\} := \{g_{n_k}\} \subseteq \{g_n\}$ so that the set

$$E_k \,:=\, \{\omega \in \Omega \,:\, |h_k(\omega) - h_{k+1}(\omega)| \ge 2^{-k}\} \quad \text{ has } \quad \mu(E_k) \le 2^{-k}\,, \quad \forall\, k \in \mathbf{N}.$$

Then $F_m := \cup_{k=m}^{\infty} E_k$ has $\mu(F_m) \le \sum_{k=m}^{\infty} \mu(E_k) \le \sum_{k=m}^{\infty} 2^{-k} = 2^{-m+1}$, and for all $k > \ell > m$, $\omega \in F_m^c$:

$$|h_\ell(\omega) - h_k(\omega)| \;\le\; \sum_{j=\ell}^{k-1} |h_{j+1}(\omega) - h_j(\omega)| \;\le\; \sum_{j=\ell}^{k-1} 2^{-j} \;\le\; 2^{-m+1}\,. \tag{10.2}$$

In other words, the sequence $\{h_k(\omega)\}_{k\in\mathbf{N}}$ is Cauchy, thus $h(\omega) := \lim_{k\to\infty} h_k(\omega)$ exists in $\mathbf{R}$, for every $\omega \in F_m^c$.

Consider $F := \cap_{m=1}^{\infty} F_m = \cap_{m=1}^{\infty} \cup_{k=m}^{\infty} E_k =: \limsup E_k$, which satisfies $\mu(F) \le \mu(F_m) \le 2^{-m+1}$ for all $m \in \mathbf{N}$, thus $\mu(F) = 0$. Therefore, the function $g = \lim_{k\to\infty} h_k \cdot \chi_{F^c}$ is well-defined, and $g = \lim_{k\to\infty} h_k$ holds $\mu-$a.e.

Now let $k \to \infty$ in (10.2), to obtain: $|h_\ell(\omega) - g(\omega)| \le 2^{-m+1}$, for all $\ell > m$, $\omega \in F_m^c$. In other words, $\{\,|h_\ell(\omega) - g(\omega)| > 2^{-m+1}\} \subseteq F_m$, for all $\ell > m$. Given any $\varepsilon > 0$, $\delta > 0$ select $m \in \mathbf{N}$ so large that $\mu(F_m) < \delta$, $\varepsilon > 2^{-m+1}$; we have then, for every $\ell > m$:

$$\mu(|h_\ell - g| > \varepsilon) \;\le\; \mu(|h_\ell - g| > 2^{-m+1}) \;\le\; \mu(F_m) \;<\delta\,.$$

In other words, the sequence $\{h_\ell\}$ converges in measure to g. But then so does the entire sequence $\{g_n\}$, since $\mu(|g_n - g| > \varepsilon) \le \mu(|g_n - h_\ell| > \varepsilon/2) + \mu(|h_\ell - g| > \varepsilon/2) < \delta$ for n, ℓ large enough.

(vi). For any $\varepsilon > 0$, $|f_n - f| \le (\varepsilon/2)$ and $|g_n - g| \le (\varepsilon/2)$ imply $|(f_n + g_n) - (f + g)| \le \varepsilon$, so that $\mu\,(\,|(f_n + g_n) - (f + g)| > \varepsilon) \le \mu\,(\,|f_n - f| > \varepsilon/2) + \mu\,(\,|g_n - g| > \varepsilon/2) \longrightarrow 0$ as $n \to \infty$.

♠ On the other hand, for any $M > 0$ we have

$$\begin{aligned}\mu\,(\,|f\,g_n - f\,g| > \varepsilon) &\le \mu\,(\,|f\,g_n - f\,g| > \varepsilon\,,\, |f| \le M\,) + \mu\,(\,|f| > M\,)\\ &\le \mu\,(\,|g_n - g| > \varepsilon/M\,) + \mu\,(\,|f| > M)\end{aligned}$$

and thus $\limsup_n \mu(\,|f\,g_n - f\,g| > \varepsilon) \,\le\, \mu(\,|f| > M)$; letting $M \to \infty$ and using the continuity of the *finite* measure μ from above (Exercise 2.2), we conclude that $\lim_n \mu(\,|f\,g_n - f\,g| > \varepsilon) = 0$. It is shown similarly that $\lim_n \mu(\,|f_n\,g - f\,g| > \varepsilon) = 0$. But now observe

$$\mu\,(\,|(f_n - f)(g_n - g)| > \varepsilon)\,\le\,\mu\left(\,|f_n - f| > \sqrt{\varepsilon}\,\right)\,+\,\mu\left(\,|g_n - g| > \sqrt{\varepsilon}\,\right)\,\longrightarrow\,0\,,$$

as $n \to \infty$. In other words $(f_n - f)(g_n - g) \longrightarrow 0$ in measure, and thus $(f_n\,g_n - f\,g) \longrightarrow 0$ in measure, in light of the previous result.

• To see how this can fail on a measure space of infinite measure, take $\Omega = (0,\infty)$ with Lebesgue measure and set $f_n(\omega) := 1 + (1/n)\,\chi_{(n,n+1]}(\omega)$, $g_n(\omega) := \omega$ for $n \in \mathbf{N}$, as well as $f(\omega) := 1$ and $g(\omega) := \omega$. The resulting sequences $\{f_n\}_{n\in\mathbf{N}}$, $\{g_n\}_{n\in\mathbf{N}}$, converge in measure to the functions f and g, respectively; indeed,

$$\{\,|f_n - f| > \varepsilon\,\} \,=\, (n, n+1) \quad \text{for} \;\; n \le (1/\varepsilon)\,, \qquad \{\,|f_n - f| > \varepsilon\,\} \,=\, \emptyset \quad \text{for} \;\; n > (1/\varepsilon)\,.$$

On the other hand, we have $f_n(\omega)g_n(\omega) - f(\omega)g(\omega) \,=\, (\omega/n)\cdot\chi_{(n,n+1]}(\omega) \,\ge\, \chi_{(n,n+1]}(\omega)$, therefore $\lambda(|f_ng_n - fg| > \varepsilon) \ge \lambda((n, n+1]) = 1$ for all $\varepsilon > 0$.

♠ If the function φ is uniformly continuous, then for every $\varepsilon > 0$ we can find $\delta > 0$, such that $|\varphi(x) - \varphi(y)| \le \varepsilon$ holds for every x, y in $\mathbf{R}$ with $|x - y| \le \delta$. Thus we get $\mu\,(\,|\varphi(f_n) - \varphi(f)| > \varepsilon) \,\le\, \mu\,(\,|f_n - f| > \delta) \,\longrightarrow\, 0$, as $n \to \infty$.

• If, on the other hand, the φ is just continuous, then for every $M > 0$ and $\varepsilon > 0$ we can find $\delta > 0$ such that $|\varphi(x) - \varphi(y)| \le \varepsilon$ holds for every x, y in $\mathbf{R}$ with $|x| \le M$, $|x - y| \le \delta$. Therefore,

$$\begin{aligned}\mu\,(\,|\varphi(f_n) - \varphi(f)| > \varepsilon) \,&\le\, \mu\,(\,|\varphi(f_n) - \varphi(f)| > \varepsilon\,,\; |f| \le M) + \mu\,(|f| > M)\\ &\le\, \mu\,(\,|f_n - f| > \delta\,,\; |f| \le M) + \mu\,(|f| > M)\;,\end{aligned}$$

thus $\limsup_{n\to\infty} \mu\,(\,|\varphi(f_n) - \varphi(f)| > \varepsilon) \,\le\, \mu\,(|f| > M)$. We conclude by letting $M \to \infty$, since $\mu(\Omega) < \infty$.

♠ Let $\{f_{n_k}\}_{k\in\mathbf{N}}$ be a subsequence of $\{f_n\}_{n\in\mathbf{N}}$, such that $\lim_k I(f_{n_k}) \,=\, \liminf_n I(f_n)$, and find a further subsequence $\{f_{n_{k_\ell}}\}_{\ell\in\mathbf{N}}$ of $\{f_{n_k}\}_{k\in\mathbf{N}}$ that converges to f, $\mu-$a.e. Then by Fatou: $I(f) \,\le\, \liminf_\ell I(f_{n_{k_\ell}}) \,=\, \lim_k I(f_{n_k}) \,=\, \liminf_n I(f_n)$.

Solution 5.6 : If $r = \infty$ then $p = q\cdot\ell$, and $\int |f|^q d\mu \,\le\, (||f||_\infty)^{q-p}\cdot\int |f|^p d\mu$, so that

$$||f||_q \,\le\, \Big(||f||_\infty\Big)^{(q-p)/q}\cdot\left(\int |f|^p d\mu\right)^{1/q} \,\le\, (||f||_\infty)^{1-\ell}\cdot(||f||_p)^{\ell}\,.$$

If $r < \infty$, use Hölder's inequality with conjugate exponents $p' = p/\ell q$, $q' = r/(1-\ell)q$ to obtain

$$\int |f|^q\,d\mu \,=\, \int |f|^{\ell q}\cdot|f|^{(1-\ell)q}\,d\mu \,\le\, \left(\int |f|^{\ell q\cdot p'}\,d\mu\right)^{1/p'}\cdot\left(\int |f|^{(1-\ell)q\cdot q'}\,d\mu\right)^{1/q'} \,=$$

$$=\left(\int |f|^p\,d\mu\right)^{\ell q/p}\cdot\left(\int |f|^r\,d\mu\right)^{(1-\ell)q/r} \,=\, (||f||_p)^{\ell q}\cdot(||f||_r)^{(1-\ell)q}\,.$$

Now take $q-$roots, to complete the argument.

Solution 5.7 : The case $q=\infty$ is easy: $\int |f|^p\,d\mu \;\leq\; (||f||_\infty)^p\cdot\mu(\Omega)$. For $q<\infty$, the Hölder inequality gives $\int |f|^p\,d\mu \;\leq\; \left(\int |f|^{pr}\,d\mu\right)^{1/r}(\mu(\Omega))^{1/s}$, where $r=(q/p)$, $(1/r)+(1/s)=1$.

Solution 5.8 : Take $\Omega=(0,\infty)$ with Lebesgue measure and, for $0<\beta<\alpha<1$, define $f(x)=x^{-\beta}$ for $0<x<1$ and $f(x)=x^{-\alpha}$ for $x\geq 1$. Then f^p is integrable on $(1,\infty)$ iff $\alpha p>1$; and it is integrable on $(0,1)$ iff $\beta p<1$. Thus, f^p is integrable on $(0,\infty)$ iff $(1/\alpha)<p<(1/\beta)$.

We see from this two reasons why f may fail to be in $\mathbf{L}^p$; either $|f|^p$ *becomes too large very rapidly near some point*, or else it *fails to decay sufficiently fast near infinity*. In the first case, the behavior of $|f|^p$ becomes worse as p increases (i.e., for $p<r$, functions in $\mathbf{L}^p$ can be locally more singular than functions in $\mathbf{L}^r$). In the second case, the behavior of $|f|^p$ becomes better as p increases (i.e., for $p<r$, functions in $\mathbf{L}^r$ can be locally more spread-out than functions in $\mathbf{L}^p$).

Solution 5.10 : (i) Let us start by recalling that Hölder's inequality $|I(fg)|\leq ||f||_p\,||g||_q$ holds as equality iff: $\alpha|f|^p=\beta|g|^q$ holds $\mu-$a.e., for some real numbers α, β with $\alpha\,\beta\neq 0$. In particular, we have $||T_f||\leq ||f||_p$, with equality if $||f||_p=0$. If $\mu(f\neq 0)>0$ and $p<\infty$, the above discussion shows that Hölder's inequality holds as equality for the function

$$g_* \;:=\; \mathrm{sgn}(f)\cdot\left(\frac{|f|}{||f||_p}\right)^{p-1}\,,$$

which also satisfies $\int |g_*|^q\,d\mu \;=\; \left(\int |f|^p\,d\mu\right)/\,(||f||_p)^p \;=\; 1$, whence $||T_f||\geq \int f g_*\,d\mu \;=\; \left(\int |f|^p\,d\mu\right)\,/\,(||f||_p)^{p-1} \;=\; ||f||_p$.

If $p=\infty$ and μ is semi-finite, we can choose for each $\varepsilon>0$ a set $F_\varepsilon\subseteq\{|f|>||f||_\infty-\varepsilon\}$ with $0<\mu(F_\varepsilon)<\infty$; then $g_\varepsilon:=(\mathrm{sgn}(f)/\mu(F_\varepsilon))\cdot\chi_{F_\varepsilon}$ satisfies $||T_f||\;\geq\;\int f g_\varepsilon\,d\mu \;=\; \left(\int_{F_\varepsilon}|f|\,d\mu\right)/\,\mu(F_\varepsilon)\;\geq\; ||f||_\infty-\varepsilon$, as well as $||g_\varepsilon||_1=\int |g_\varepsilon|\,d\mu \;=\; \left(\int \chi_{F_\varepsilon}\,d\mu\right)/\mu(F_\varepsilon)\;=\;1$.

(ii) From Hölder's inequality, it is clear that $N(f)\;\leq\;||f||_p$, so we need to prove the reverse inequality $N(f)\geq ||f||_p$.

If $p=\infty$, suppose that the set $A=\{|f|>N(f)+\varepsilon\}$ has positive measure for some $\varepsilon>0$, and choose $B\subset A$ with $0<\mu(B)<\infty$. Then for the simple (and vanishing outside a set of finite measure) function

$$\hat{g}:=\mathrm{sgn}(f)\,\chi_B\,/\,\mu(B) \qquad \text{we have} \quad ||\hat{g}||_1=1 \quad \text{and} \qquad \int f\hat{g}\,d\mu \;=\; \frac{1}{\mu(B)}\int |f|\,d\mu\;\geq\; N(f)+\varepsilon\,,$$

contradicting the definition on $N(f)$. Therefore $\mu(|f|>N(f)+\varepsilon)=0$, whence also $N(f)+\varepsilon\geq ||f||_p$, holds for all $\varepsilon>0$.

If $1\leq p<\infty$ and in addition μ is $\sigma-$finite (we shall deal with this case only), let us write $\Omega=\cup_{n=1}^\infty\Omega_n$ for an increasing sequence $\{\Omega\}_{n=1}^\infty$ of sets in $\mathcal{F}$ with $0<\mu(\Omega_n)<\infty$, and consider a sequence $\{\varphi_n\}_{n=1}^\infty$ of simple functions such that $\lim_n\varphi_n=f$ pointwise and $|\varphi_n|\leq|f|$, $\forall n\in\mathbf{N}$. Then $f_n:=\varphi_n\cdot\chi_{\Omega_n}\in\mathcal{S}_0$, and $\lim_n f_n=f$ pointwise, $|f_n|\leq|f|$ for all $n\in\mathbf{N}$. Setting as before

$$g_n \;:=\; \mathrm{sgn}(f)\cdot\left(\frac{|f_n|}{||f_n||_p}\right)^{p-1}\,, \quad \text{we have } ||g_n||_q=1\,,\ \int |f_n g_n|\,d\mu=||f_n||_p \ \text{ and } \ |f_n\,g_n|\leq|f g_n|=f g_n\,,$$

and by Fatou's lemma:

$$||f||_p\;\leq\;\liminf_{n\to\infty}||f_n||_p\;=\;\liminf_{n\to\infty}\int |f_n g_n|\,d\mu\;\leq\;\liminf_{n\to\infty}\int |f g_n|\,d\mu$$

$$= \liminf_{n\to\infty} \int f g_n \, d\mu \;\le\; N(f)\,.$$

Solution 5.11 : For $f \in \mathbf{L}^p$, choose a sequence $\{f_n\}_{n=1}^\infty$ of simple functions (e.g., $f_n = \sum_{n=1}^N \alpha_n \chi_{E_n}$ with $\alpha_n \neq 0$ and $\{E_n\}$ disjoint) such that $|f_n| \le |f|$ and $f_n \longrightarrow f$, $\mu-$a.e.; recall Exercise 2.6. Then $f_n \in \mathbf{L}^p$ since $p < \infty$, $|f_n - f| \le 2|f| \in \mathbf{L}^p$, and $f_n \longrightarrow f$ in $\mathbf{L}^p$ by the Dominated Convergence Theorem. Moreover, $\sum_{n=1}^N |\alpha_n|^p \mu(E_n) = \Big(||f_n||_p\Big)^p < \infty$ implies $\mu(E_n) < \infty$.

Solution 5.12 : Suppose that f is continuous and has compact support; then it is also uniformly continuous, and $\lim_{x\to 0} \big(\sup_{y\in\mathbf{R}} |f_x(y) - f(y)|\big) = 0$. But in this case both f_x and f are supported on a common compact set for $|x| \le 1$, so we also have

$$\lim_{x\to 0} \int_{\mathbf{R}} |f_x(y) - f(y)|^p \, dy \;=\; 0\,.$$

For $f \in \mathbf{L}^p(\mathbf{R})$ and arbitrary $\varepsilon > 0$, we choose a continuous function g with compact support and $||f - g||_p < \varepsilon/3$. Then we have also $||f_x - g_x||_p = ||f - g||_p < \varepsilon/3$, and $||g_x - g||_p < \varepsilon/3$ for $|x|$ sufficiently small, so that we obtain from the triangle inequality

$$||f_x - f||_p \;\le\; ||f_x - g_x||_p + ||g_x - g||_p + ||g - f||_p < \varepsilon\,.$$

Solution 5.13 : For $r < p < \infty$ we have:

$$\int_\Omega |f|^p \, d\mu \;=\; \int_\Omega |f|^r \, |f|^{p-r} \, d\mu \;\le\; \Big(||f||_\infty\Big)^{p-r} \cdot \int_\Omega |f|^r \, d\mu \;<\; \infty\,,$$

so $f \in \mathbf{L}^p$. Also from this: $||f||_p \;\le\; \Big(||f||_\infty\Big)^{1-(r/p)} \cdot \Big(||f||_r\Big)^{r/p}$, and letting $p \to \infty$ we obtain: $\limsup_{p\to\infty} ||f||_p \;\le\; ||f||_\infty$.

On the other hand, for any $a > 0$ with $\mu(|f| > a) > 0$ we have from Čebyšev's inequality: $\int_\Omega |f|^p \, d\mu \;\ge\; a^p \cdot \mu(|f| > a) > 0$, thus $||f||_p \;\ge\; a \cdot \Big(\mu(|f| > a)\Big)^{1/p}$. Sending $p \to \infty$ we obtain $\liminf_{p\to\infty} ||f||_p \;\ge\; a$, and taking supremum over a yields $\liminf_{p\to\infty} ||f||_p \;\ge\; ||f||_\infty$.

Solution 5.14 : It is clear that we have: $\frac{d}{du}|f + ug|^p \;=\; \frac{d}{du}\big((f + ug)^2\big)^{p/2} \;=\; p\, g\, (f + ug)\,\big((f + ug)^2\big)^{(p/2)-1} \;=\; p\,(f + ug) g\, \big|f + ug\big|^{p-2}$, so that

$$\lim_{u\to 0} \frac{1}{u} \Big(|f + ug|^p - |f|^p\Big) \;=\; p\,|f|^{p-2}\, f\, g\,.$$

The question is whether we can pass the limit under the integral sign in

$$\frac{F(f + ug) - F(f)}{u} \;=\; \int_\Omega \frac{|f + ug|^p - |f|^p}{u} \, d\mu\,.$$

To see that we can, observe

$$|f + ug|^p \;=\; |(1 - u) f + u(f + g)|^p \;\le\; (1 - u)\,|f|^p \;+\; u\,|f + ug|^p\,, \qquad 0 < u \le 1$$

from the convexity of $x \mapsto |x|^p$, so that $|f + ug|^p - |f|^p \le u\big(|f + g|^p - |f|^p\big)$. A similar argument gives $|f + ug|^p - |f|^p \le u\big(|f|^p - |f - g|^p\big)$, for $-1 \le u < 0$. Therefore,

$$|f|^p - |f - g|^p \;\le\; \frac{1}{u} \Big(|f + ug|^p - |f|^p\Big) \;\le\; |f + g|^p - |f|^p\,, \qquad u \in [-1, 1] \setminus \{0\}\,.$$

The functions $f,\ f \pm g$ are in $\mathbf{L}^p$, so the Dominated Convergence Theorem allows us to conclude.

Solution 5.15 : We shall concentrate on the case $1 < p < 2$, and try to prove (5.8) written in the form

$$\int |f+g|^p \, d\mu + \int |f-g|^p \, d\mu \geq (A+B)^p - (A-B)^p , \quad \text{assuming } A := ||f||_p \geq ||g||_p =: B \quad (5.8)'$$

without loss of generality. To see this, observe that for given $R \in (0,1]$ the function

$$F_R(r) := \alpha(r) + \beta(r)\, R^p , \quad 0 \leq r \leq 1$$

with $\alpha(r) := (1+r)^{p-1} + (1-r)^{p-1}$, $\beta(r) := \big[(1+r)^{p-1} - (1-r)^{p-1}\big]\, r^{1-p}$, attains its maximum $F_R(R) = \alpha(R) + \beta(R)R^p = (1+R)^p + (1-R)^p$ ar $r = R$. Therefore, we have

$$\alpha(r) \cdot A^p + \beta(r) \cdot B^p \leq (A+B)^p + (A-B)^p \qquad \text{for } 0 \leq r \leq 1, \ 0 < B \leq A, \tag{10.3}$$

with equality for $r = B/A$. In view of this last inequality, to prove (5.8)′ it suffices to show

$$\int |f+g|^p \, d\mu + \int |f-g|^p \, d\mu \geq \alpha(r) \cdot \int |f|^p \, d\mu + \beta(r) \cdot \int |f|^p \, d\mu ,$$

or even $(\varphi+\gamma)^p + |\varphi-\gamma|^p \geq \alpha(r) \cdot \varphi^p + \beta(r) \cdot \gamma^p$ for $\gamma > 0$, $\varphi > 0$, $0 \leq r \leq 1$. But with $\varphi \geq \gamma$, this follows from (9.3); whereas with $\varphi < \gamma$, the inequality (10.3) gives

$$(\varphi+\gamma)^p + (\gamma-\varphi)^p \geq \alpha(r) \cdot \gamma^p + \beta(r) \cdot \varphi^p \geq \alpha(r) \cdot \varphi^p + \beta(r) \cdot \gamma^p ,$$

because $\alpha(r) \cdot \rho^p + \beta(r) \geq \alpha(r) + \beta(r) \cdot \rho^p$ if $\rho > 1 \geq r \geq 0$.

Once (5.8) has been established, (5.9) follows if one replaces f by $f+g$, and g by $f-g$. A similar argument deals with the case $p > 2$.

Solution 5.16 : Let us concentrate on $1 < p < 2$, $f \equiv 0$. Take a minimizing sequence $\{g_n\} \subset \mathcal{G}$, with $||g_n||_p \downarrow \delta$ as $n \to \infty$. We shall try to show that this is a Cauchy sequence, so that $\big|\, ||g_n||_p - ||g_*||_p \,\big| \leq ||g_n - g_*||_p \to 0$ as $n \to \infty$ for some $g_* \in \mathcal{G}$; this will also show $||g_*||_p = \delta$.

To see all this, observe that convexity and the triangle inequality give

$$\delta \leq \Big|\Big| \frac{1}{2}(g_n + g_m) \Big|\Big|_p \leq \frac{1}{2}\big(||g_n||_p + ||g_m||_p\big) \longrightarrow \delta \qquad \text{as } n, m \to \infty ,$$

so that $||g_n + g_m||_p \to 2$ as $n, m \to \infty$.

Suppose for a moment that $||g_n - g_m||_p \to 0$ as $n, m \to \infty$ *fails;* in other words, that there exists an $\varepsilon > 0$ such that $||g_n - g_m||_p \geq \varepsilon$ holds for infinitely many m and n in $\mathbf{N}$. Back in (5.9) of Exercise 5.15, this implies

$$|2\delta + \varepsilon|^p + |2\delta - \varepsilon|^p \leq 2^{p+1}\, \delta^p ,$$

contradicting the *strict* convexity of $x \mapsto |x|^p$.

Thus $\{g_n\} \subset \mathcal{G}$ is a Cauchy sequence, that converges to some $g_* \in \mathcal{G}$ in $\mathbf{L}^p$. For any $g \in \mathcal{G}$, $0 \leq u \leq 1$ we have $g_u := (1-u)g_* + ug \in \mathcal{G}$ by convexity, and the function

$$u \mapsto F(u) := \int_\Omega |(1-u)g_* + ug|^p \, d\mu = \big(||g_u||_p\big)^p$$

has $F(u) \geq \delta = F(0)$. From Exercise 5.14, $F(\cdot)$ is differentiable at $u = 0$, and thus $F'(0) = p \int_\Omega |g_*|^p \, g_* \big(g - g_* \big) \, d\mu \geq 0$.

Solution 5.18 : (i) On $\Omega = [0,1]$ with Lebesgue measure λ, look at $\xi_n = n \, \chi_{(0,1/n)}$, $n \in \mathbf{N}$ and observe that $I(\xi_n) = 1$ holds for every $n \in \mathbf{N}$, so we have boundedness in $\mathbf{L}^1$. On the other hand,

$$\{ \xi_n > \kappa \} = \emptyset \quad \text{for} \ \ \kappa \geq n \,, \qquad \{ \xi_n > \kappa \} = (0, 1/n) \quad \text{for} \ \ 0 < \kappa < n \,,$$

thus $\sup_{n \in \mathbf{N}} \int_{\{\xi_n > \kappa\}} \xi_n \, d\lambda = 1$ for every $\ell \in (0, \infty)$ and uniform integrability fails.

(ii) On the same probability space as before, consider now the family of functions $f_{A,n} = n \, \chi_A$, $\lambda(A) = 1/n^2$ $(A \in \mathcal{B}([0,1]), n \in \mathbf{N})$. Clearly, there is no $g \in \mathbf{L}^1$ with $0 \leq f_{A,n} \leq g$ a.e. for every (A,n). Yet

$$\{ f_{A,n} > \kappa \} = \emptyset \quad \text{for} \ \ \kappa \geq n \,, \qquad \{ f_{A,n} > \kappa \} = A \quad \text{for} \ \ 0 < \kappa < n \,,$$

thus

$$\int_{\{f_{A,n} > \kappa\}} f_{A,n} \, d\lambda = n \cdot \lambda(A) \, \chi_{(\kappa, \infty)}(n) = (1/n) \, \chi_{(\kappa,\infty)}(n) \leq \frac{1}{\kappa} \,, \qquad \forall \ (A,n), \, \kappa > 0 \,,$$

so $\sup_{(A,n)} \int_{\{f_{A,n} > \kappa\}} \xi_n \, d\lambda \leq (1/\kappa) \to 0$ as $\kappa \to \infty$, and uniform integrability holds.

Solution 6.1 : Let us justify Remark 6.1 first. For any $\alpha \in A$ and $E_\alpha \in \mathcal{F}_\alpha$, we have $\pi_\alpha^{-1} = \{\omega \in \Omega \,|\, \omega(\alpha) \in E_\alpha\} = \prod_{\beta \in A} E'_\beta$, where $E'_\beta = \Omega_\beta$ for $\beta \neq \alpha$, and $E'_\beta = E_\alpha$ for $\beta \neq \alpha$. Therefore $\mathcal{C} \subseteq \mathcal{R}$, $\mathcal{F} = \sigma(\mathcal{C}) \subseteq \sigma(\mathcal{R})$. On the other hand, $\prod_{\alpha \in A} E_\alpha = \{ \omega \in \Omega \,|\, \omega(\alpha) \in E_\alpha \,, \ \forall \, \alpha \in A \} = \bigcap_{\alpha \in A} \pi^{-1}(E_\alpha) \in \sigma(\mathcal{C}) = \mathcal{F}$ if A is countable, so $\mathcal{R} \subseteq \mathcal{F}$ and $\sigma(\mathcal{R}) \subseteq \mathcal{F}$.

Returning to Exercise 6.1, we need to show $\mathcal{F} = \sigma(\mathcal{C}) \subseteq \sigma(\mathcal{C}')$. For any given $\alpha \in A$, the class $\mathcal{M}_\alpha := \{E \in \Omega_\alpha \,|\, \pi_\alpha^{-1}(E) \in \sigma(\mathcal{C}')\}$ is a $\sigma-$algebra that contains $\mathcal{E}_\alpha$; thus $\mathcal{F}_\alpha \subseteq \mathcal{M}_\alpha$, i.e., $\pi_\alpha^{-1}(E) \in \sigma(\mathcal{C}')$, $\forall \, E \in \mathcal{F}_\alpha$, $\alpha \in A$, or equivalently $\mathcal{C} \subseteq \sigma(\mathcal{C}')$, which implies $\mathcal{F} = \sigma(\mathcal{C}) \subseteq \sigma(\mathcal{C}')$. The second claim follows by the argument used to justify Remark 6.1.

Solution 6.2 : From Exercise 6.1 we have $\bigotimes_{j=1}^n \mathcal{B}(\Omega_j) = \sigma(\mathcal{C}')$ where $\mathcal{C}' = \{\pi_j^{-1}(O_j)\,;\, O_j$ open in Ω_j, $1 \leq j \leq n \}$ and $\pi_j^{-1}(O_j) = \prod_{k=1}^n E_k$ (with $E_k = \Omega_k$, $k \neq j$ and $E_k = O_j$, $j = k$) is open in Ω; therefore, $\mathcal{C}' \subseteq \mathcal{B}(\Omega)$, $\bigotimes_{j=1}^n \mathcal{B}(\Omega_j) = \sigma(\mathcal{C}') \subseteq \mathcal{B}(\Omega)$.

Now let each Ω_j have a countable, dense subset D_j, and denote by $\mathcal{S}_j$ the countable collection of rectangles with rational sides, centered at the points of D_j. Then every open rectangle in Ω_j is a (countable) union of rectangles in $\mathcal{S}_j$, so that $\sigma(\mathcal{S}_j) = \mathcal{B}(\Omega_j)$, and thus $\sigma(\{\prod_{j=1}^n B_j \,;\, B_j \in \mathcal{S}_j \,, \ \forall \, j = 1, \cdots, n\}) = \bigotimes_{j=1}^n \mathcal{B}(\Omega_j)$ from Exercise 6.1. Finally, observe that $\mathcal{B}(\Omega) = \sigma(\{\prod_{j=1}^n B_j \,;\, B_j \in \mathcal{S}_j \,, \ \forall \, j = 1, \cdots, n\})$ (since $\prod_{j=1}^d D_j$ is countable and dense in Ω, and the rectangles in Ω are products of rectangles in the Ω_j 's).

Solution 6.3 : The second part follows directly from Example 6.1, with $K(x,y) \equiv g(x-y)$ and $\nu = \lambda =$ Lebesgue measure on $\mathcal{B}(\mathbf{R}^d)$. For the third part, note that Young's inequality guarantees that the convolution $(f * g)(\xi)$ is well-defined, for $\lambda-$a.e. $\xi \in \mathbf{R}^d$, and that we can apply the Tonelli-Fubini theorems in tandem to justify changing the order of integration in

$$\begin{aligned} (\widehat{f * g})(\xi) &= \int_{\mathbf{R}^d} e^{\,i\langle \xi, x\rangle} (f * g)(x) \, dx = \int_{\mathbf{R}^d} e^{\,i\langle \xi, x\rangle} \left(\int_{\mathbf{R}^d} f(x-y) g(y) \, dy \right) dx \\ &= \int_{\mathbf{R}^d} e^{\,i\langle \xi, y\rangle} \left(\int_{\mathbf{R}^d} f(x-y) \, e^{\,i\langle \xi, x-y\rangle} \, dx \right) g(y) \, dy = \widehat{f}(\xi) \int_{\mathbf{R}^d} e^{\,i\langle \xi, y\rangle} g(y) \, dy = \widehat{f}(\xi) \, \widehat{g}(\xi) \,. \end{aligned}$$

More precisely, the applicability of Fubini's theorem is justified by

$$\int_{\mathbf{R}^d} |e^{\,i\langle \xi, x\rangle}| \left(\int_{\mathbf{R}^d} |f(x-y)|\,|g(y)|\,dy \right) dx \;=\; \int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} |f(x-y)|\,dx \right) |g(y)|\,dy$$
$$= \;||f||_1 \int_{\mathbf{R}^d} |g(y)|\,dy \;=\; ||f||_1 \cdot ||g||_1 < \infty \,,$$

itself a consequence of Tonelli's theorem and the integrability of f and g.

Solution 6.4 : From Tonelli's theorem we have that $\int_{[0,\infty)} \mu(g>u)\,d\nu(u)$ is equal to

$$\int_{[0,\infty)} \left(\int_\Omega \chi_{(u,\infty)}(g(\omega))\,d\mu(\omega) \right) d\nu(u) \;=\; \int_\Omega \left(\int_{[0,\infty)} \chi_{[0,g(\omega))}\,d\nu(u) \right) d\mu(\omega)\,,$$

which is equal to $\int_\Omega \left(\int_{[0,g(\omega))} (u)\,d\nu(u) \right) d\mu(\omega) \;=\; \int_\Omega N(g(\omega))\,d\mu(\omega)\,.$

Solution 6.5 : We have $(\mathcal{P}\delta_0)(x) = |x|$ and

$$(\mathcal{P}\mu)(x) \;=\; \int_{(-\infty,x]} (x-y)\,d\mu(y) \;+\; \int_{(x,\infty)} (y-x)\,d\mu(y) \;=\; xF(x) \;+\; 2\int_{(x,\infty)} y\,d\mu(y) - x(1-F(x))$$

since $\int_{\mathbf{R}} y\,d\mu(y) = 0$. Therefore, for $x>0$ the expression

$$\big(\mathcal{P}\mu - \mathcal{P}\delta_0\big)(x) \;=\; 2\int_{(x,\infty)} (y-x)\,d\mu(y) \;=\; 2\int_x^\infty (1-F(y))\,dy \;\geq\; 0$$

tends to zero as $x\to\infty$; whereas for $x<0$ the expression

$$\big(\mathcal{P}\mu - \mathcal{P}\delta_0\big)(x) \;=\; 2\int_{(-\infty,x]} (x-y)\,d\mu(y) \;=\; 2\int_{-\infty}^x F(y)\,dy \;\geq\; 0$$

tends to zero as $x\to-\infty$. Finally, by Tonelli

$$\int_{-\infty}^\infty \big(\mathcal{P}\mu - \mathcal{P}\delta_0\big)(x) \;=\; 2\int_0^\infty \left(\int_{(x,\infty)} (y-x)\,d\mu(y) \right) dx$$
$$+\,2\int_{-\infty}^0 \left(\int_{(-\infty,x]} (x-y)\,d\mu(y) \right) dx \;=\; \int_{\mathbf{R}} y^2\,d\mu(y)\,.$$

Solution 6.6 : It is clear that we can assume $I(F(g))<\infty$. If (6.12) holds for the pair (f,g), then it holds also for $(f\wedge n, g)$, for each $n>0$; and if we can establish (6.13) for each of these latter pairs, then we have established it also for (f,g), by letting $n\to\infty$ and appealing to the Monotone Convergence Theorem. Thus, without loss of generality, we may assume $I(F(f))<\infty$ as well.

Pick $\gamma>0$ such that $F(x/\beta) \geq \gamma\,F(x)$ holds for every $x>0$, and integrate both sides of (6.9) with respect to $dF(\lambda)$; from the Tonelli-Fubini theorems (recall also Exercise 6.4), this gives

$$\psi(\delta)\cdot I(F(f)) \geq \int_0^\infty \mu\left(\frac{g}{\delta} \leq \lambda < \frac{f}{\beta} \right) dF(\lambda) = I\left(\Big(F(f/\beta) - F(g/\delta) \Big)^+ \right)$$
$$\geq I\left(F(f/\beta)\right) - I\left(F(g/\delta)\right) \geq \gamma \cdot I\left(F(f)\right) - I\left(F(g/\delta)\right)\,;$$

thus $(\gamma-\psi(\delta))\cdot I\,(F(f))\le I\,(F(g/\delta))$. If we select $\delta\in(0,1)$ so small, that $\gamma-\psi(\delta)>(\gamma/2)$, and then pick $\zeta>0$ so that $F(x/\delta)\le\zeta\cdot F(x)$ holds for every $x>0$, then we obtain $(\gamma/2)\cdot I\,(F(f))\le \zeta\cdot I\,(F(g/\delta))$; this is (6.13) with $C=(2\zeta)/\gamma$, independent of f and g.

Solution 6.7 : Take $\Omega_1=\Omega_2=\mathbf{R}$ endowed with the σ-algebra $\mathcal{L}$ of *Lebesgue*-measurable sets, and with the (completed) Lebesgue measure $\bar\lambda$; fix $a\in\mathbf{R}$ and a non-Lebesgue-measurable set $\Xi\notin\mathcal{L}$; recall (4.3) and Proposition A.1, Appendix A. Now set $E_1=\{a\},\ E_2=\Xi,\ E=E_1\times E_2$; then E is a subset of $\{a\}\times\mathbf{R}$ which has zero $(\bar\lambda\otimes\bar\lambda)-$measure. But E does *not* belong to the product σ-algebra, because its section $E_{\omega_1}=\Xi$ at $\omega_1=a$ is not (Lebesgue-) measurable.

To remedy this situation as indicated, proceed as follows. Take an $\bar{\mathcal{F}}-$measurable function $f:\Omega\to\mathbf{R}$ with $f=0$, $\bar\mu-$a.e.; argue that its sections f_{ω_1}, f_{ω_2} are integrable and $\int_{\Omega_2} f_{\omega_1}\,d\mu_2=\int_{\Omega_1} f_{\omega_2}\,d\mu_1=0$, for μ_1-a.e. ω_1, μ_2-a.e. ω_2 (here the completeness of the component spaces is crucial). Now use Exercise 3.6.

Solution 6.8 : We shall discuss the one-dimensional case $d=1$ only. Let us start by observing that $\int\varphi_\varepsilon(x)\,dx=1$, which implies

$$\big(f*\varphi_\varepsilon\big)(x)-f(x)\;=\;\int\big[f(x-y)-f(x)\big]\varphi_\varepsilon(y)\,dy\;=\;\int\big[f(x-\varepsilon y)-f(x)\big]\varphi(y)\,dy\,.$$

(i) Recalling Exercise 5.12 and its notation, along with the Minkowski inequality for integrals (Proposition 6.2), we obtain:

$$\big|\big|(f*\varphi_\varepsilon)\;-\;f\big|\big|_p\;\le\;\int\big|\big|f_{-\varepsilon y}\;-\;f\big|\big|_p\,|\varphi(y)|\,dy\;\longrightarrow\;0\qquad\text{as }\;\varepsilon\downarrow 0$$

by Dominated Convergence, because $\big|\big|f_{-\varepsilon y}\;-\;f\big|\big|_p\;\le\;2\,\big|\big|f\big|\big|_p<\infty$ and $\big|\big|f_{-\varepsilon y}\;-\;f\big|\big|_p\longrightarrow 0$ as $\varepsilon\downarrow 0$, for each $y\in\mathbf{R}$.
(ii) For $f\in\mathbf{L}^\infty(\mathbf{R})$ uniformly continuous on a set B, and for any given $\delta>0$, let us select a bounded set F so that $\int_{\mathbf{R}\setminus F}|\varphi(x)|\,dx<\delta$; then

$$\sup_{x\in B}\big|(f*\varphi_\varepsilon)(x)\;-\;f(x)\big|\;\le\;2\delta\,||f||_\infty+\sup_{x\in B,\,y\in F}\big|f(x-\varepsilon y)-f(x)\big|\cdot\int_F|\varphi(y)|\,dy\;\longrightarrow\;2\delta\,||f||_\infty$$

as $\varepsilon\downarrow 0$, and the result follows from the arbitrariness of $\delta>0$.
(iii) For every $\varphi\in C^\infty_\downarrow(\mathbf{R})$ and bounded $F\subset\mathbf{R}$, we have

$$\sup_{x\in F}\big|\big(D^m\varphi\big)(x-y)\big|\;\le\;C_{m,F}\,\big(1+|y|\;\big)^{-2},\qquad y\in\mathbf{R}\,,$$

for every $m\in\mathbf{N}_0$. The function $y\mapsto\big(1+|y|\;\big)^{-2}$ is in $\mathbf{L}^q(\mathbf{R})$, where $(1/p)+(1/q)=1$, and thus the integral

$$\Big[f*\big(D^m\varphi\big)\Big](x)\;=\;\int_{\mathbf{R}}\big(D^m\varphi\big)(x-y)\,f(y)\,dy$$

converges absolutely and uniformly on bounded sets. Then from Exercise 5.9(ii) we can exchange differentiation and integration, and arrive at (6.14).

Solution 6.9 : Choose $\varphi\in C^\infty_*(\mathbf{R})$ with $\int\varphi(x)\,dx=1$, and introduce the functions φ_ε as in Exercise 6.8, for $\varepsilon>0$. If $f\in\mathbf{L}^p(\mathbf{R})$ has compact support, then so does $(f*\varphi_\varepsilon)$ (Exercise 6.3(i)), and we know from Exercise 6.8 that $(f*\varphi_\varepsilon) \in C^\infty(\mathbf{R})$. In other words, $(f*\varphi_\varepsilon) \in C_*^\infty(\mathbf{R})$, and from Exercise 6.8 we deduce that $\big|\big|(f*\varphi_\varepsilon) - f\big|\big|_p \longrightarrow 0$, as $\varepsilon \downarrow 0$. But the set of functions $f \in \mathbf{L}^p(\mathbf{R})$ with compact support is dense in $\mathbf{L}^p(\mathbf{R})$, and this completes the argument.

Solution 7.1 : Set $Z_n := X - X_n$ for $n \in \mathbf{N}$; since $\int Z_n \, d\lambda = \mu(\Omega) - \mu_n(\Omega) = 0$, we have $\mu(E) - \mu_n(E) = \int_E Z_n \, d\lambda = -\int_{E^c} Z_n \, d\lambda$ as well as

$$2\left|\mu(E) - \mu_n(E)\right| = 2\left|\int_E Z_n \, d\lambda\right| = \left|\int_E Z_n \, d\lambda\right| + \left|\int_{E^c} Z_n \, d\lambda\right| \le \int |Z_n| \, d\lambda = 2\int Z_n^+ \, d\lambda$$

for any $E \in \mathcal{F}$, with equality for $E = \{Z_n \ge 0\}$. This means

$$2\,\big|\big|\mu_n - \mu\big|\big| = \int |Z_n| \, d\lambda = 2\int Z_n^+ \, d\lambda \, .$$

Now $0 \le Z_n^+ \le X$ and $Z_n^+ \to 0$ hold $\lambda-$a.e., which implies $\int Z_n^+ \, d\lambda \longrightarrow 0$ as $n \to \infty$, by the Dominated Convergence Theorem.

Solution 7.3: Assume $\mu << \nu$, let $X = d\mu/d\nu$ and denote integration with respect to ν by I. Then, using the identity in (7.7), it suffices to show $2\,I(\,X \log X\,) \ge \big(I(|X-1|)\big)^2$.

Define $Y = X - 1$, and observe the elementary inequality

$$(1+y)\cdot\log(1+y) \ge y + \frac{y^2}{2}\,\frac{1}{1+(y/3)}\,, \quad \text{for } y \ge -1\,.$$

In conjunction with the simple observation $I(Y) = 0$ this gives

$$2\,I(\,X \log X\,) = 2\,I\Big((1+Y)\cdot\log(1+Y) - Y\Big) \ge I\left(\frac{Y^2}{1+(Y/3)}\right)$$

and from Cauchy-Schwarz we see that $I\left(\frac{Y^2}{1+(Y/3)}\right) = I\left(\frac{Y^2}{1+(Y/3)}\right)\cdot I\left(1+(Y/3)\right)$ dominates

$$\left(I\left(\frac{|Y|}{\sqrt{1+(Y/3)}}\cdot\sqrt{1+(Y/3)}\right)\right)^2 = \Big(I(|X-1|)\Big)^2 .$$

Solution 7.4 : Note $H(\mu_\alpha|\nu) = \int \xi_\alpha \left(\log(\xi_\alpha)\right)^+ d\nu = \int f(\xi_\alpha)\, d\nu$ where $\xi_\alpha := d\mu_\alpha/d\nu$ and $f(x) := x\left(\log x\right)^+$. The result follows from Exercise 5.17 (ii).

Solution 7.5: (Atar & Zeitouni (1997)) There is nothing to prove if λ and μ are not comparable; so let us assume they are, and set

$$\mathcal{B} := \{\, B \in \mathcal{F} \,|\, \lambda(B) \ge \mu(B) > 0 \,\}\,, \qquad \mathcal{C} := \{\, C \in \mathcal{F} \,|\, \lambda(C) < \mu(C) \,\}\,.$$

Note that $\mathcal{B}$ is nonempty, and that if $\mathcal{C}$ is empty then $\mu = \lambda$ and once again there is nothing to prove.

Thus we take $\mathcal{B} \neq \emptyset$, $\mathcal{C} \neq \emptyset$ from now on, and note

$$1 \le \frac{\lambda(B)}{\mu(B)} \le \frac{\lambda(B)}{\mu(B)}\cdot\frac{\mu(C)}{\lambda(C)} \le e^{\,h(\lambda,\mu)}\,, \qquad \forall\ B \in \mathcal{B},\ C \in \mathcal{C}\,.$$

This implies

$$0 \le \lambda(B) - \mu(B) \le \mu(B)\left(e^{h(\lambda,\mu)} - 1\right), \qquad \forall\ B \in \mathcal{B}$$

$$0 < \mu(C) - \lambda(C) \le \lambda(C)\left(e^{h(\lambda,\mu)} - 1\right), \qquad \forall\ C \in \mathcal{C}$$

and

$$2 \cdot ||\lambda - \mu|| = \sup_{B \in \mathcal{B}} \Big[\big(\lambda(B) - \mu(B)\big) \vee \big(\mu(B^c) - \lambda(B^c)\big) \Big] \le e^{h(\lambda,\mu)} - 1 .$$

Solution 7.7: Take $\varepsilon = 1$ in the definition of absolute continuity, and let N be the greatest integer not exceeding $1 + (b-a)/\delta$, For any division $a = x_0 < x_1 < \cdots < x_n = b$, we can collect (by inserting more subdivision points, if necessary) the intervals (x_{i-1}, x_i) into at most N groups of consecutive intervals, whose lengths sum up to at most δ in each group. Then the sum $\sum_i |f(x_i) - f(x_{i-1})|$ is at most one over each group, so the total variation of f on $[a,b]$ is at most N.

Solution 8.1: Write $\Omega = \cup_{n=1}^{\infty} E_n$ for some increasing sequence $\{E_n\} \subseteq \mathcal{F}$ with $0 < \mu(E_n) < \infty$, and identify $\mathbf{L}^r_n(\mu) \equiv \mathbf{L}^r(E_n, \mu)$ with the set of functions in $\mathbf{L}^r(\mu) \equiv \mathbf{L}^r(\Omega, \mu)$ which vanish outside the set E_n. From (8.5), there exists for each $n \in \mathbf{N}$ a function $f_n \in \mathbf{L}^p_n(\mu)$ with $\Phi(g) = \int_\Omega f_n g\, d\mu$, $\forall\, g \in \mathbf{L}^q_n(\mu)$ and $||f_n||_p = \big|\big|\Phi\big|_{\mathbf{L}^q_n(\mu)}\big|\big| \le ||\Phi|| < \infty$.

This f_n is unique modulo $\mu-$a.e. equivalence, so $f_n = f_m$ holds $\mu-$a.e. on E_n for $m > n$, and we can define $f : \Omega \to \mathbf{R}$ consistently by setting $f := f_n$ on E_n. We have then $||f||_p = \lim_n ||f_n||_p \le ||\Phi|| < \infty$ by monotone convergence, and $g_n := g\,\chi_{E_n} \longrightarrow g$ in $\mathbf{L}^q(\mu)$ by dominated convergence for every $g \in \mathbf{L}^q(\mu)$. It follows that

$$\Phi(g) = \lim_n \Phi(g\,\chi_{E_n}) = \lim_n \int_\Omega f_n g\, d\mu = \lim_n \int_\Omega f g_n\, d\mu = \int_\Omega f g\, d\mu .$$

Solution 8.3: (i) The first comparison is clear. The rather obvious set inclusion $\{|f+g| > 2u\} \subseteq \{|f| > u\} \cup \{|g| > u\}$ leads to the second comparison. And integrating $|f|^p = \int_0^\infty \chi_{\{|f|^p > \xi\}}\, d\xi$ with respect to μ, gives

$$\int_\Omega |f|^p\, d\mu = \int_0^\infty \mu\left(|f|^p > \xi\right) d\xi = p \int_0^\infty u^{p-1}\, \lambda_f(u)\, du$$

with the help of Tonelli and the change of variable $\xi = u^p$.

(ii) For $\alpha \ne 0$ we have $\lambda_{\alpha f}(u) = \lambda_f(u/|\alpha|)$, which leads to the first claim. The second is an easy consequence of the comparisons

$$\sup_{u>0}\big((2u)^p \lambda_{f+g}(2u)\big) \le 2^p \cdot \sup_{u>0}\Big(u^p\big(\lambda_f(u) + \lambda_g(u)\big)\Big) \le 2^p \cdot \left(\sup_{u>0}\big(u^p \lambda_f(u)\big) + \sup_{u>0}\big(u^p \lambda_g(u)\big)\right) .$$

The comparisoon $[f]_p \le ||f||_p$ is a direct consequence of the Čebyšev inequality.

Solution 9.1: The idea is to apply the Recurrence Theorem 9.1 to all powers of T. Fix an arbitrary $k \in \mathbf{N}$ and let F_k be the set of points in E that never return to E under successive actions of T^k; by Theorem 9.1 we have $\mu(F_k) = 0$. Now for every $\omega \in E \setminus (F_1 \cup F_2 \cup \cdots)$ we have $T^k(\omega) \in E$ for some $k \in \mathbf{N}$, since $\omega \in E \setminus F_1$; as well as $T^{km}(\omega) \in E$ for some $m \in \mathbf{N}$, since $\omega \in E \setminus F_k$. It remains to repeat inductively this (already twice repeated) argument.

To prove (9.1) for a.e. $\omega \in \{f > 0\}$, consider the set $E_k = \{\omega \in \Omega \,|\, f(\omega) > 1/k\}$. The Recurrence Theorem 9.1 implies that for a.e. $\omega \in E_k$ we have: $T^j(\omega) \in E_k$ for infinitely many $j \in \mathbf{N}$, thus $\sum_{j \in \mathbf{N}} f(T^j(\omega)) = \infty$. Therefore, this property holds for a.e. $\omega \in \cup_k E_k = \{f > 0\}$.

Solution 9.2: (ii) If there are no non-constant invariant functions, it is clear (just by considering indicator functions) that there cannot possibly be any non-trivial invariant sets – and thus that T is ergodic.

Now suppose that T is ergodic and that $f : \Omega \to \mathbf{R}$ is measurable and invariant, and try to show that f is constant a.e. If $C^{n,k} = \{k2^{-n} \le f < (k+1)2^{-n}\}$, then the invariance of f implies that of $C^{n,k}$; and, for each $n \in \mathbf{N}$, the ergodicity of T now gives $\mu(C^{n,k}) = 0$ for all but one $k \in \mathbf{N}$. Now take the intersection (over n) of all the 'large' ones among the sets $C^{n,k}$.

Solution 9.3: This T is clearly measure-preserving. If c is a root of unity, then $f(\omega) = \omega^n$ is measurable, $T-$invariant and non-constant.

If c is not a root of unity, then the mappings $\omega \mapsto f(\omega) = \omega^n$, $n \in \mathbf{Z}$ form a complete orthonormal system in $\mathbf{L}^2$. Thus every $f \in \mathbf{L}^2$ can be written as $f = \sum_{n \in \mathbf{Z}} a_n f_n$, where the series is understood to converge in $\mathbf{L}^2$. With $(Uf)(\omega) := f(T(\omega))$ we observe $Uf_n = c^n f_n$, and so $Uf_n = \sum_{n \in \mathbf{Z}} a_n c^n f_n$. Now if f is invariant we must have $a_n = a_n c^n$ for all integers, thus $a_n = 0$ for all $n \neq 0$, and consequently $f \equiv a_0$. In other words, every invariant function in $\mathbf{L}^2$ is a constant, so T is ergodic.

Solution 9.4: Let $f : \Omega \to \mathbf{R}$ be square-integrable; then the Fourier series $\sum_{n \in \mathbf{Z}} c_n e^{2\pi i n \omega}$ with $\sum_{n \in \mathbf{Z}} |c_n|^2 < \infty$ of $f(\omega)$ converges in $\mathbf{L}^2$, and because T is measure-preserving we have

$$c_n = \int_\Omega f(\omega)\, e^{2\pi i n \omega}\, d\omega = \int_\Omega f(T(\omega))\, e^{2\pi i n T(\omega)}\, d\omega = e^{2\pi i n \xi} \int_\Omega f(T(\omega))\, e^{2\pi i n T(\omega)}\, d\omega$$

$$= e^{2\pi i n \xi} \int_\Omega f(\omega)\, e^{2\pi i n \omega}\, d\omega = c_n \cdot e^{2\pi i n \xi}, \quad \forall \ n \in \mathbf{N}.$$

If ξ is irrational, then we have $e^{2\pi i n \xi} \neq 1$, so $c_n = 0$, for every $n \in \mathbf{N}$; thus f is then a.e. equal to a constant, and T is ergodic by Exercise 9.2(iii).

If $\xi = k/m$ for integers k and m, then the set $A = \cup_{k=0}^{2m-1} \{\omega \in \Omega \,:\, k/(2m) \le \omega < (k+1)/(2m)\}$ is clearly invariant, but has Lebesgue measure 1/2.

Solution 9.6: (a) With $A \in \mathcal{I}$, that is, $T^{-1}A = A \mod. \mu$, we have $T^{-k}A = A \mod. \mu$, thus $\mu(A \cap T^{-k}A) = \mu(A)$, for every $k \in \mathbf{N}$. Therefore, taking $B = A$ in the weak mixing property (9.6), we obtain $\mu(A) = \mu^2(A)$, so $\mu(A) = 0$ or 1. In other words, T is ergodic.

• If T is ergodic, then Corollary 9.1 applied to $f = \chi_B$, $B \in \mathcal{F}$ gives $\lim_{n \to \infty} (1/n) \sum_{k=0}^{n-1} \chi_{T^{-k}B} = \mu(B)$ a.e. Integrate both sides over $A \in \mathcal{F}$ and use the dominated (or even bounded) convergence theorem, to obtain (9.6).

(b) Let us assume that T is ergodic, and try to show (9.7) (the other implication is now easy). Because T is measure-preserving, the mapping $\varphi \mapsto \varphi \circ T$ is an isometry on $\mathbf{L}^2(\mu)$, and for given $f \in \mathbf{L}^2(\mu)$ the set of averages $\{(1/n) \sum_{k=0}^{n-1} f \circ T^n\}_{n \in \mathbf{N}}$ belongs to a closed ball in this Hilbert space. Such a ball is compact in the weak topology of the space, so the above sequence of averages will converge weakly in the space (i.e., (9.7) will hold for any $g \in \mathbf{L}^2(\mu)$) once it has been established that the set in question has a unique limit point.

Any such limit point, however, is a T-invariant function, thus constant by ergodicity. Since

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_\Omega f(T^n(\omega))\, d\mu(\omega) \;=\; \int_\Omega f(\omega)\, d\mu(\omega)\,,$$

this constant must be $\int_\Omega f(\omega)\, d\mu(\omega)$.