1.2. LIMITS AND INTEGRALS

The first important property of the class of measurable functions is that it is *closed under limits.* In fact, if $\{f_n\}_{n\in\mathbf{N}}$ is a sequence of real-valued functions, we can define the pointwise limit-superior and limit-inferior by

$$\limsup_{n\to\infty} f_n(\omega) := \inf_k \left(\sup_{n\geq k} f_n(\omega)\right), \quad \liminf_{n\to\infty} f_n(\omega) := \sup_k \left(\inf_{n\geq k} f_n(\omega)\right), \tag{2.1}$$

respectively. These quantities exist for every $\omega \in \Omega$, possibly with values $+\infty$ or $-\infty$, since the sequences $\left\{\sup_{n\geq k} f_n(\omega)\right\}_{k\in\mathbf{N}}$ and $\{\inf_{n\geq k} f_n(\omega)\}_{k\in\mathbf{N}}$ are monotone (decreasing and increasing, respectively).

For the purposes of this section it is convenient then to consider functions $f : \Omega \to [-\infty, +\infty]$ with values in the extended real line. Such a function f is said to be *measurable*, if $f^{-1}((a, +\infty]) \in \mathcal{F}$ for every $a \in \mathbf{R}$ (equivalently, when $f^{-1}([-\infty, a)) \in \mathcal{F}$ for every $a \in \mathbf{R}$). Obviously, this definition agrees with the earlier one if f is finite everywhere.

Let us assume, therefore, that $\{f_n\}_{n\in\mathbf{N}}$ is a sequence of measurable functions with values in the extended real line $[-\infty, +\infty]$. Then for any $k \in \mathbf{N}$ we have

$$\begin{aligned} \left(\sup_{n\geq k} f_n\right)^{-1}((a,\infty]) &= \cup_{n\geq k} f_n^{-1}((a,\infty]) \\ \left(\inf_{n\geq k} f_n\right)^{-1}([-\infty,a)) &= \cup_{n\geq k} f_n^{-1}([-\infty,a)), \end{aligned} \tag{2.2}$$

so that the supremum and the infimum of a sequence of measurable functions are themselves measurable functions. Iterating the argument we deduce that the limit-inferior and the limit-superior are also measurable. In particular, if the sequence $\{f_n\}_{n\in\mathbf{N}}$ converges pointwise, its limit is measurable. As for the integrals of the functions in the sequence $\{f_n\}_{n\in\mathbf{N}}$, we have the following three fundamental results, the pillars of measure theory.

We shall denote throughout by $\mathbf{L}^0$ the space of measurable functions $f : \Omega \to \mathbf{R}$; by $\mathbf{L}^1$ its subspace of integrable functions; by $\mathbf{L}^0_+$ the space of measurable functions $f : \Omega \to [0, \infty)$; and by $\mathbf{L}_+$ the space of measurable functions $f : \Omega \to [0, \infty]$. The integral $I(f)$ of $f \in \mathbf{L}_+$ is defined as in (1.2), (1.3).

2.1 THEOREM : B. LEVI's MONOTONE CONVERGENCE. *Suppose that the sequence $\{f_n\}_{n\in\mathbf{N}} \subset \mathbf{L}_+$ is monotone increasing, in the sense that $f_n \leq f_{n+1}$ holds pointwise for all $n \in \mathbf{N}$. Then*

$$\lim_{n\to\infty} \int_\Omega f_n \, d\mu = \int_\Omega \left(\lim_{n\to\infty} f_n\right) d\mu \,. \tag{2.3}$$

Note that neither side of (2.3) is assumed to be finite.

We cannot dispense in this result with the assumption of monotonicity; for instance, consider the functions $f_n = n\,\chi_{(0,1/n)}$, $n \in \mathbf{N}$ on $\Omega = (0,1]$ with Lebesgue measure on its Borel sets. Observe that we have $I(f_n) = 1$ for all $n \in \mathbf{N}$, that the sequence $\{f_n\}_{n\in\mathbf{N}}$ *is not monotone*, and that $f := \lim_n f_n \equiv 0$ pointwise, thus $I(f) = 0$.

Note, however, that $I(f) \le \lim_n I(f_n)$ holds in this example, echoing a more general result known as "Fatou's Lemma".

2.2 THEOREM : FATOU's LEMMA. *For any sequence* $\{f_n\}_{n\in\mathbf{N}} \subset \mathbf{L}_+$ *we have*

$$\int_\Omega \big(\liminf_{n\to\infty} f_n\big)\, d\mu \;\le\; \liminf_{n\to\infty} \int_\Omega f_n\, d\mu\,. \tag{2.4}$$

2.3 THEOREM : LEBESGUE's DOMINATED CONVERGENCE. *Let* $\{f_n\}_{n\in\mathbf{N}} \subset \mathbf{L}^0$ *be any sequence of measurable functions with* $|f_n| \le g$ *valid pointwise for all* $n \in \mathbf{N}$ *, where* $g \in \mathbf{L}^1$ *is an integrable function. If the sequence* $\{f_n\}_{n\in\mathbf{N}}$ *converges pointwise, then the limit is integrable and (2.3) holds.*

We begin by establishing Theorem 2.1. As a preparatory step, let us consider the case where each f_n is the indicator function of a measurable set E_n , namely, $f_n = \chi_{E_n}$ for all $n \in \mathbf{N}$. Then Theorem 2.1 is equivalent to the following property of measure, known as **continuity from below:**

$$\lim_{n\to\infty} \mu(E_n) = \mu(\cup_{n\in\mathbf{N}} E_n) \tag{2.5}$$

whenever $\{E_n\}_{n\in\mathbf{N}} \subseteq \mathcal{F}$ is a monotone increasing sequence of measurable sets, namely with $E_n \subseteq E_{n+1}$ for every $n \in \mathbf{N}$.

- To establish the property (2.5) let us express each E_n as a finite disjoint union $E_n = \cup_{k=1}^n F_k$ with $F_1 = E_1$, and $F_k = E_k \setminus E_{k-1}$ as well as $\mu(F_k) = \mu(E_k) - \mu(E_{k-1})$ for $k \ge 2$; in particular, we have then $\cup_{n\in\mathbf{N}} E_n = \cup_{k\in\mathbf{N}} F_k$ ("cutting in disjoint slices"). The countable additivity of the measure μ implies

$$\mu(\cup_{n\in\mathbf{N}} E_n) = \mu(\cup_{k\in\mathbf{N}} F_k) = \sum_{k\in\mathbf{N}} \mu(F_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(F_k) = \lim_{n\to\infty} \mu(E_n).$$

- Returning to the proof of Theorem 2.1, let us recall the monotonicity property $(1.3)'$ of the integral $I(f)$, $f \in \mathbf{L}_+$, which gives

$$\int_\Omega f_n\, d\mu \le \int_\Omega f_{n+1}\, d\mu \le \int_\Omega f\, d\mu\,, \quad \forall\, n \in \mathbf{N} \tag{2.6}$$

since $f_n \leq f_{n+1} \leq \lim_{n\to\infty} f_n =: f$. The sequence $\left\{\int_\Omega f_n \, d\mu\right\}_{n\in\mathbf{N}}$ is increasing, and hence has a limit in $[0,\infty]$. In view of (2.6) we obtain

$$\lim_{n\to\infty} \int_\Omega f_n \, d\mu \;\leq\; \int_\Omega f \, d\mu \,.$$

To prove the reverse inequality, fix momentarily a number $c \in (0,1)$ as well as a simple function g satisfying $0 \leq g \leq f$, and set

$$E_n := \{\omega \in \Omega \,|\, f_n(\omega) \geq cg(\omega)\} \equiv \{f_n \geq cg\}.$$

Then we have $E_n \subseteq E_{n+1}$ and $\chi_{E_n} \leq \chi_{E_{n+1}}$ pointwise for all $n \in \mathbf{N}$, as well as $\Omega = \cup_{n\in\mathbf{N}} E_n$. The homogeneity and monotonicity properties $(1.2)'$, $(1.3)'$ of the integral imply

$$c \int_{E_n} g \, d\mu \,=\, c \cdot I\left(g\chi_{E_n}\right) = I\left(c \cdot g\chi_{E_n}\right) \equiv \int_{E_n} c\, g \, d\mu \;\leq\; \int_{E_n} f_n \, d\mu \;\leq\; \int_\Omega f_n \, d\mu.$$

The limit as $n \to \infty$ of the sequence of integrals $\left\{I\left(g\chi_{E_n}\right)\right\}_{n\in\mathbf{N}}$ exists, and equals

$$\lim_{n\to\infty} \int_{E_n} g \, d\mu = \lim_{n\to\infty} \sum_{k=1}^{K} y_k \cdot \mu\left(g^{-1}(\{y_k\}) \cap E_n\right) = \sum_{k=1}^{K} y_k \cdot \mu\left(g^{-1}(\{y_k\})\right) = \int_\Omega g \, d\mu$$

thanks to the continuity-from-below property of (2.5); here $g(\Omega) = \{y_1, \cdots, y_K\}$ is the range of the simple function g. Thus we obtain

$$c \int_\Omega g \, d\mu \;\leq\; \lim_{n\to\infty} \int_\Omega f_n \, d\mu \,.$$

Taking the supremum over all simple functions g satisfying $0 \leq g \leq f$, and then over all $c \in (0,1)$, yields

$$\int_\Omega f \, d\mu \;\leq\; \lim_{n\to\infty} \int_\Omega f_n \, d\mu$$

and hence Theorem 2.1.

• Let us consider Fatou's lemma (2.4) next. Recall that $\liminf_{n\to\infty} f_n$ is the pointwise limit of the monotone increasing sequence of functions $h_k := \inf_{n\geq k} f_n$, $k \in \mathbf{N}$. From $h_k \leq f_k$, the monotonicity of the integral gives $I(h_k) \leq I(f_k)$; from Theorem 2.1 we obtain then

$$\int_\Omega (\lim_{k\to\infty} h_k) \, d\mu \;=\; \lim_{k\to\infty} \int_\Omega h_k \, d\mu \;\leq\; \liminf_{k\to\infty} \int_\Omega f_k \, d\mu \,, \tag{2.7}$$

and Fatou's lemma is proved.

• We turn to the proof of Theorem 2.3. It is instructive to see first, why *the requirement that all $|f_n|$ be dominated by a fixed integrable function g cannot be removed.*

Consider again the case where each f_n is the indicator function of a set $E_n \in \mathcal{F}$: namely, $f_n = \chi_{E_n}$. Assume that $E_{n+1} \subseteq E_n$, and $\cap_{n\in\mathbf{N}} E_n = \emptyset$. Thus $f_n \downarrow 0$ pointwise, and $\int_\Omega f_n \, d\mu = \mu(E_n)$. It is easy, however, to find sets $\{E_n\}$ satisfying the previous conditions, and yet $\mu(E_n) = \infty$ for all n; take, for example, $E_n = (n, \infty)$ with $\mu = \lambda \equiv$ Lebesgue measure.

The difficulty is easily identified if we try to adapt the treatment of the special case in the proof of (2.3) to the B. Levi Monotone Convergence Theorem. Indeed, for any decreasing sequence $\{E_n\}_{n\in\mathbf{N}} \subseteq \mathcal{F}$ we can write E_1 as a countable union $\cup_{k\in\mathbf{N}} F_k$ of disjoint subsets with $F_k := E_k \setminus E_{k+1}$ and thus $\mu(F_k) = \mu(E_k) - \mu(E_{k+1})$, so that by countable additivity

$$\mu(E_1) = \sum_{k\in\mathbf{N}} \mu(F_k) = \lim_{n\to\infty} \sum_{k=1}^{n-1} \mu(F_k) = \lim_{n\to\infty} \Big(\mu(E_1) - \mu(E_n) \Big). \tag{2.8}$$

The additional hypothesis $\mu(E_1) < \infty$ would allow us to subtract $\mu(E_1)$ from both sides and conclude with the **continuity from above** property $\lim_{n\to\infty} \mu(E_n) = 0$ **under the hypothesis** $\mu(E_1) < \infty$ (see Exercise 2.2 in the same vein). In our special context, *this hypothesis is exactly the same as the requirement that all $|f_n|$ be dominated by an integrable function.*

The step of subtracting $\mu(E_1)$ in the above simple case will require the **linearity property of the integral**

$$\underbrace{\int_\Omega (\alpha f + \beta \phi) \, d\mu = \alpha \int_\Omega f \, d\mu + \beta \int_\Omega \phi \, d\mu}\,, \quad \text{for any real constants } \alpha, \beta \tag{2.9}$$

and any (integrable) real-valued functions f, ϕ in $\mathbf{L}^1$. This property is certainly not evident from our definition (1.3) of the integral. (Nor, for that matter, is it completely evident that *linear combinations of measurable functions are measurable;* cf. Exercise 1.9.)

However, assuming both these statements for the moment, we can prove the Lebesgue Dominated Convergence Theorem along the lines of the previous example. Since $|f_n| \le g$, we also have $|f| \le g$ for $f := \lim_{n\to\infty} f_n$. The sequence $\{2g - |f - f_n|\}_{n\in\mathbf{N}} \subset \mathbf{L}^0_+$ converges to $2g$, so by Fatou's Lemma

$$\int_\Omega 2g \, d\mu \le \liminf_{n\to\infty} \int_\Omega (2g - |f - f_n|) \, d\mu = \int_\Omega 2g \, d\mu - \limsup_{n\to\infty} \int_\Omega |f - f_n| \, d\mu \,.$$

It follows that $\limsup_{n\to\infty} \int_\Omega |f - f_n| \, d\mu \le 0$, and from Exercise 2.4 (i) we get

$$\left| \int_\Omega f \, d\mu - \int_\Omega f_n \, d\mu \right| = \left| \int_\Omega (f - f_n) \, d\mu \right| \le \int_\Omega |f - f_n| \, d\mu \longrightarrow 0, \quad \text{as } n \to \infty \,. \qquad \diamond$$

• We come now to the issue of **linearity of the integral**. Let us begin by recalling that *the sum and product of a finite number of measurable functions is measurable* (Exercise 1.9), and check the linearity of the integral for *simple functions*. Let g, h be simple functions with values $\{y_1, \cdots, y_M\}$ and $\{z_1, \cdots, z_N\}$ respectively; we set $E_j := g^{-1}(\{y_j\})$, $F_k := h^{-1}(\{z_k\})$, and assume $E_i \cap E_j = \emptyset = F_k \cap F_\ell$ for $i \neq j$, $k \neq \ell$ without loss of generality. Then $\alpha s_1 + \beta s_2$ is a simple function for any real constants α and β, with value $\alpha y_j + \beta z_k$ on the set $E_j \cap F_k$. Hence, by the finite additivity of the measure μ, we obtain

$$\int_\Omega (\alpha g + \beta h)\, d\mu = \sum_{j=1}^{M} \sum_{k=1}^{N} (\alpha y_j + \beta z_k) \mu(E_j \cap F_k) = \alpha \sum_{j=1}^{M} y_j \mu(E_j) + \beta \sum_{k=1}^{N} z_k \mu(F_k)$$

$$= \alpha \int_\Omega g\, d\mu + \beta \int_\Omega h\, d\mu\,, \tag{2.10}$$

as was to be shown.

• The next step in the proof of the linearity of the integral requires us to replace the supremum in the definition (1.3) by a more convenient limit. The key ingredient is the fact that **for any non-negative, measurable function $f \in \mathbf{L}_+$, there exists a monotone increasing sequence of non-negative, simple functions $\{g_n\}_{n\in\mathbf{N}}$ converging pointwise to f**, namely:

$$0 \leq g_n(\omega) \leq g_{n+1}(\omega) \to f(\omega) \qquad \text{as } n \to \infty\,, \quad \text{for every } \omega \in \Omega\,. \tag{2.11}$$

These simple functions $\{g_n\}$ are easily constructed by partitioning the *range* of f (rather than its domain, as we are accustomed to, from Riemann integration...) according to dyadic rationals, as follows. For each $n \in \mathbf{N}$, we define $g_n := n$ on the set $f^{-1}([n, \infty])$, and

$$g_n := \frac{k}{2^n} \qquad \text{on} \qquad f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right)\,, \qquad \text{for } k = 0, \cdots, n2^n - 1 \tag{2.12}$$

on the set $f^{-1}([0, n))$. The situation is messier to write down than to visualize, so the reader is urged to draw a picture.

Since $[k\, 2^{-n}, (k+1)\, 2^{-n}) = \cup_{\ell=2k}^{2k+1} \left[\ell\, 2^{-(n+1)}, (\ell+1)\, 2^{-(n+1)}\right)$, the only possible values of g_{n+1} on this set are $2k\, 2^{-(n+1)}$ and $(2k+1)\, 2^{-(n+1)}$, both of which are at least as big as the value $k\, 2^{-n}$ of g_n on this set. Thus, $g_{n+1} \geq g_n$ on $f^{-1}([0, n))$. Similarly, on $f^{-1}([n, \infty])$ we have $g_n = n \leq g_{n+1}$, so that $\{g_n\}_{n\in\mathbf{N}}$ is a monotone increasing sequence of simple functions. Furthermore, $f - (1/2^n) \leq g_n \leq f$ on the set $f^{-1}([0, n))$. Any point $\omega \in \Omega$ with $f(\omega) < \infty$ belongs to some $f^{-1}([0, n))$ for n large enough, and thus satisfies

$g_n(\omega) \to f(\omega)$. But at points $\omega \in \Omega$ with $f(\omega) = \infty$ we have $g_n(\omega) = n$ for all n, and thus again $g_n(\omega) \to f(\omega)$, establishing (2.12).

In view of the Monotone Convergence Theorem, it follows then from (2.11) that

$$I(f) \equiv \int_\Omega f \, d\mu = \lim_{n\to\infty} \int_\Omega g_n \, d\mu \, . \tag{2.13}$$

• Let now f and ϕ be (*non-negative*, measurable) functions in $\mathbf{L}_+$, and α and β be positive constants. Let $\{g_n\}_{n\in\mathbf{N}}$ and $\{h_n\}_{n\in\mathbf{N}}$ be sequences of non-negative, simple functions, monotonically increasing to f and ϕ, respectively.

Then $\{\alpha g_n + \beta h_n\}_{n\in\mathbf{N}}$ is a sequence of nonnegative, simple functions, monotonically increasing to the (nonnegative, measurable) function $\alpha f + \beta\phi \in \mathbf{L}_+$. The linearity property (2.9) follows now from (2.10) and the Monotone Convergence Theorem.

• Finally, let us note that $\mathbf{L}^1$ is a real vector space: if f and ϕ are *real-valued* integrable functions, then so is $h := \alpha f + \beta\phi$ for any real constants α and β. Recalling $I(cf) = c\,I(f)$ of $(1.2)'''$, valid for every $c \in \mathbf{R}$, it becomes clear that we need verify (2.9) only for $\alpha = \beta = 1$. We do this by separating f and ϕ into their positive and negative parts $f^\pm$ and $\phi^\pm$. The details are straightforward; indeed, with $h = f + \phi$ we have

$$h^+ - h^- = f^+ - f^- + \phi^+ - \phi^- \, , \qquad \text{so that} \qquad h^+ + f^- + \phi^- = h^- + f^+ + \phi^+ \, ;$$

from what has already been shown, we obtain $I(h^+) + I(f^-) + I(\phi^-) = I(h^-) + I(f^+) + I(\phi^+)$, so that $I(h) = I(f) + I(\phi)$. The proof of (2.9) is complete.

2.1 EXERCISE: Čebyšev Inequality. For any measurable function f, we have

$$\mu(|f| \geq a) \equiv \mu\left(\{\omega \in \Omega : |f(\omega)| \geq a\}\right) \leq \frac{1}{a^p} \int_\Omega |f|^p \, d\mu \, , \qquad \forall \;\; a > 0, \; p > 0 \, . \tag{2.14}$$

Note that if $\int_\Omega |f|^p \, d\mu < \infty$ for some $p > 0$, then the set $\{f \neq 0\}$ is σ-finite.

2.2 EXERCISE: Continuity of measure from above. Show that we have

$$\lim_{n\to\infty} \mu(E_n) = \mu\big(\cap_{n\in\mathbf{N}} E_n\big) \, , \tag{2.15}$$

whenever $\{E_n\}_{n\in\mathbf{N}} \subseteq \mathcal{F}$ is a monotone decreasing sequence of measurable sets: $E_{n+1} \subseteq E_n$ for every $n \in \mathbf{N}$, provided $\mu(E_m) < \infty$ for some $m \in \mathbf{N}$.

2.3 EXERCISE: For any $f \in \mathbf{L}_+$ and $\{f_n\}_{n\in\mathbf{N}} \subset \mathbf{L}_+$ we have:

(i) $I(f) = 0 \iff \mu(f \neq 0) = 0$;

(ii) $I\left(\sum_{n\in\mathbf{N}} f_n\right) = \sum_{n\in\mathbf{N}} I(f_n)$;

(iii) $I(f) = \sup_{\substack{\varphi \in \mathbf{L}_+ \\ 0 \le \varphi \le f}} I(\varphi)$;

(iv) if $I(f) < \infty$ then $E = \{\omega \in \Omega \,|\, f(\omega) = \infty\}$ is a null set, and on the set $F = \{\omega \in \Omega \,|\, f(\omega) > 0\}$ the measure μ is σ-finite;

(v) if μ, ν are two measures with $\mu(E) \le \nu(E)$, $\forall\, E \in \mathcal{F}$, then $\int_\Omega f\, d\mu \le \int_\Omega f\, d\nu$ for every $f \in \mathbf{L}_+$;

(vi) the mapping $\nu : \mathcal{F} \to [0, \infty]$ defined by $\nu(E) := \int_E f\, d\mu$ is a measure, and we have $\int_\Omega g\, d\nu = \int_\Omega fg\, d\mu$ for every $g \in \mathbf{L}_+$.

2.4 EXERCISE: For any integrable, real-valued functions f, g and $\{f_n\}_{n \in \mathbf{N}}$ we have:

(i) $|I(f)| \le I(|f|)$;

(ii) $I(|f + g|) \le I(|g|) + I(|f|)$;

(iii) $\int_E f\, d\mu = \int_E g\, d\mu$, $\forall\, E \in \mathcal{F} \iff I(|f - g|) = 0 \iff \mu(f \ne g) = 0$;

(iv) if $\sum_{n \in \mathbf{N}} I(|f_n|) < \infty$, then $\sum_{n \in \mathbf{N}} f_n$ converges μ–a.e. to an integrable function, with $I\left(\sum_{n \in \mathbf{N}} f_n\right) = \sum_{n \in \mathbf{N}} I(f_n)$.

2.5 EXERCISE: For any sequence $\{E_n\}_{n \in \mathbf{N}}$ of measurable sets, we have:

(i) : $\mu\left(\liminf_{n \to \infty} E_n\right) \le \liminf_{n \to \infty} \mu(E_n)$;

(ii) : $\mu\left(\limsup_{n \to \infty} E_n\right) \ge \limsup_{n \to \infty} \mu(E_n)$, if $\mu\left(\cup_{n \ge k} E_n\right) < \infty$ for some $k \in \mathbf{N}$;

(iii) : $\mu\left(\limsup_{n \to \infty} E_n\right) = 0$ if $\sum_{n \in \mathbf{N}} \mu(E_n) < \infty$.

The property (iii) is referred to as the *First Borel-Cantelli Lemma.* We recall here the definitions of the limit-inferior and the limit-superior

$$\liminf_{n \to \infty} E_n \equiv \{E_n\,,\ \text{for all but finitely-many } n \in \mathbf{N}\} := \cup_{k \in \mathbf{N}} \cap_{n \ge k} E_n\,,$$

$$\limsup_{n \to \infty} E_n \equiv \{E_n\,,\ \text{for infinitely many } n \in \mathbf{N}\} \equiv \{E_n\,,\ \text{i.o.}\} := \cap_{k \in \mathbf{N}} \cup_{n \ge k} E_n\,,$$

for a sequence of sets, where the qualifier *"i.o."* stands for "infinitely often".

2.6 EXERCISE: Approximating Measurable by Simple Functions. For every measurable function $f : \Omega \to \mathbf{R}$ there exists a sequence $\{g_n\}_{n \in \mathbf{N}}$ of simple functions with $|g_1| \le |g_2| \le \ldots \le |f|$; with $g_n \longrightarrow f$ pointwise; and with $\sup_{\omega \in E} |g_n(\omega) - f(\omega)| \longrightarrow 0$ as $n \to \infty$, for any set $E \in \mathcal{F}$ on which f is bounded.

2.7 EXERCISE: Composition of Measurable Functions, Change of Variable. Let $(\Omega, \mathcal{F})$ and $(X, \mathcal{G})$ be two measurable spaces, suppose the mapping $T : \Omega \to X$ is measurable (that is, $T^{-1}G \in \mathcal{F}$ holds for every $G \in \mathcal{G}$), and for a given measure μ on $\mathcal{F}$ we define a measure $m = \mu T^{-1}$ on $\mathcal{G}$ by $m(G) := \mu\big(T^{-1}G\big)$, $G \in \mathcal{G}$.

Consider also a $\mathcal{G}-$measurable function $f : X \to \mathbf{R}$.

(i) Show that the composition $fT : \Omega \to \mathbf{R}$ defined by $(fT)(\omega) := f(T(\omega))$ is $\mathcal{F}-$measurable.

(ii) If f is non-negative, establish the change-of-variable formula

$$\int_{T^{-1}G} fT \, d\mu = \int_G f \, dm, \qquad \forall \; G \in \mathcal{G}.$$

(iii) A little more generally, show that f is integrable with respect to m if and only if fT is integrable with respect to μ, and that under this condition the change-of-variable formula is valid again.

2.8 Exercise: An Extended Dominated Convergence Theorem. Let $\{f_n\}$, $\{g_n\}$ be functions in $\mathbf{L}^1$, such that

$$|f_n| \le g_n \quad (\forall \; n \in \mathbf{N}), \qquad \lim_{n\to\infty} f_n = f, \qquad \lim_{n\to\infty} g_n = g$$

hold a.e. for some $f \in \mathbf{L}^0$ and $g \in \mathbf{L}^1$. Assume also that $\lim_{n\to\infty} I(g_n) = I(g)$.

Show that we have then $f \in \mathbf{L}^1$, as well as

$$\lim_{n\to\infty} I(f_n) = I(f), \qquad \lim_{n\to\infty} I(|f_n - f|) = 0.$$

2.9 EXERCISE: Let f, g be real-valued, measurable functions of $(\Omega, \mathcal{F})$.

(i) Observe that the collection $\sigma(f) := f^{-1}(\mathcal{B}(\mathbf{R}))$ is a σ-algebra, and $\sigma(f) \subseteq \mathcal{F}$. It is called the *$\sigma$-algebra generated by f*, and is the smallest σ-algebra with respect to which f is measurable.

(ii) For $\sigma(g) \subseteq \sigma(f)$, it is necessary and sufficient that there exist a Borel-measurable function $h : \mathbf{R} \to \mathbf{R}$ such that $g(\omega) = h(f(\omega))$, $\omega \in \Omega$.