1.3. OUTER MEASURE

A systematic way of constructing measures according to certain ‘primitive’ requirements is provided by the Carathéodory-Hahn theory. This approach consists of

(i) the Carathéodory Theorem 3.1, which identifies a σ-algebra, and a measure μ on it, from any *outer measure* μ^* (an outer measure is defined on all subsets of Ω, but is only countably *sub*-additive); of

(ii) Theorem 3.2 below, which shows how an outer measure μ^* arises from any set-function $\nu : \mathcal{E} \to [0, \infty]$ defined on a family $\mathcal{E}$ of subsets of Ω with $\emptyset \in \mathcal{E}$, $\Omega \in \mathcal{E}$ and $\nu(\emptyset) = 0$; and of

(iii) the Hahn Extension Theorem 3.3. This provides conditions under which μ^* is an extension of ν and shows that the extension is then essentially unique.

The Lebesgue-Stieltjes measures outlined in Examples (e)-(f) of Section 1.1 are obtained, for instance, by choosing $\mathcal{E}$ to be the algebra that contains all finite, disjoint unions of half-open intervals $(a, b]$, and requiring $\nu((a, b\,]) = F(b) - F(a)$.

3.1 Definition : Outer Measure. Let Ω be a given space. A set-function μ^* defined on all subsets of Ω with values in $[0, \infty]$ is said to be an *outer measure*, if

(i) $\mu^*(\emptyset) = 0$;

(ii) $\mu^*(E) \le \mu^*(F)$ if $E \subseteq F$;

(iii) $\mu^*(\cup_{n \in \mathbf{N}} E_n) \le \sum_{n \in \mathbf{N}} \mu^*(E_n)$ for any sequence $\{E_n\}_{n \in \mathbf{N}}$ of subsets of Ω.
(This is the so-called *countable sub-additivity property.*)

For instance, take $\Omega = \mathbf{N}$ and consider $\mu^*(A) = 0$ if $A = \emptyset$ and $\mu^*(A) = 1$ if $A \ne \emptyset$, $A \subseteq \mathbf{N}$; this recipe defines an outer measure which is clearly not additive, even finitely. A systematic way of constructing outer measures is outlined in Theorem 3.2 below, and justifies the terminology.

3.1 THEOREM : THE CARATHÉODORY CONSTRUCTION. *Let μ^* be an outer measure on a space Ω, and consider the family $\mathcal{M}$ of subsets E of Ω which satisfy the condition*

$$\mu^*(A) \ge \mu^*(A \cap E) + \mu^*(A \cap E^c), \qquad \forall \;\; A \subseteq \Omega. \tag{3.1}$$

Then $\mathcal{M}$ is a σ-algebra, and the restriction $\overline{\mu} = \mu^*\big|_{\mathcal{M}}$ of μ^* to $\mathcal{M}$ is a measure.

Clearly, the inequality (3.1) need be checked only for sets $A \subseteq \Omega$ with $\mu^*(A) < \infty$. Because of the sub-additivity property (iii) of Definition 3.1, the reverse of (3.1) always holds; therefore, this condition is equivalent to

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c), \qquad \forall \;\; A \subseteq \Omega. \tag{3.1$'$}$$

Note also that for an outer-measure μ^* there can exist sets A and E with $\mu^*(A) < \mu^*(A \cap E) + \mu^*(A \cap E^c)$; for instance, in the example immediately following Definition 3.1, take $A = \{n-1, n, n+1\}$ and $E = \{n, n+1, \cdots\}$ for some $n \geq 2$.

Proof of Theorem 3.1 : Evidently $\mathcal{M}$ contains the empty set, and is closed under complementation (since (3.1) is symmetric in E, E^c).

We begin by showing that $\mathcal{M}$ is *closed under finite unions*. With E, F arbitrary elements of $\mathcal{M}$, and A any subset of Ω, we write $A \cap (E \cup F) = (A \cap E \cap F) \cup (A \cap E^c \cap F) \cup (A \cap E \cap F^c)$, so the subadditivity of μ^* implies

$$\begin{aligned}\mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^c) \leq \mu^*(A \cap E \cap F) + \mu^*(A \cap E^c \cap F) \\ + \mu^*(A \cap E \cap F^c) + \mu^*(A \cap E^c \cap F^c). \qquad (3.2)\end{aligned}$$

But the condition (3.1) for E implies

$$\mu^*(A \cap E \cap F) + \mu^*(A \cap E^c \cap F) \leq \mu^*(A \cap F), \quad \mu^*(A \cap E \cap F^c) + \mu^*(A \cap E^c \cap F^c) \leq \mu^*(A \cap F^c),$$

respectively. Substituting these in (3.2) and using (3.1) yet again, this time for F, we arrive at

$$\mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^c) \leq \mu^*(A \cap F) + \mu^*(A \cap F^c) \leq \mu^*(A).$$

We conclude that $E \cup F$ is in $\mathcal{M}$, and thus $\mathcal{M}$ is closed under pairwise, thus also under finite, unions.

• To show that $\mathcal{M}$ is closed also under countable unions, we consider next a sequence of sets $\{E_n\}_{n \in \mathbf{N}} \subseteq \mathcal{M}$. Their union can be written as the union of *pairwise disjoint* sets, namely

$$\cup_{n \in \mathbf{N}} E_n = \cup_{n \in \mathbf{N}} F_n \in \mathcal{M}, \quad \text{where} \quad F_n := E_n \setminus \left(\cup_{k=1}^{n-1} E_k\right) \qquad (3.3)$$

for $n \geq 2$ and $F_1 := E_1$. Each set F_n is in $\mathcal{M}$, in view of the fact that $\mathcal{M}$ is closed under finite unions and under complementation.

Thus, we may assume that the $\{E_n\}_{n \in \mathbf{N}}$ are pairwise disjoint to begin with. It follows then from $(3.1)'$ that for any $A \subseteq \Omega$ with $\mu^*(A) < \infty$, we have

$$\begin{aligned}\mu^*(A \cap (\cup_{k=1}^n E_k)) &= \mu^*(A \cap (\cup_{k=1}^n E_k) \cap E_n) + \mu^*(A \cap (\cup_{k=1}^n E_k) \cap E_n^c) \\ &= \mu^*(A \cap E_n) + \mu^*(A \cap (\cup_{k=1}^{n-1} E_k)), \qquad \forall\, n \in \mathbf{N}\end{aligned}$$

and thus

$$\mu^*(A \cap (\cup_{k=1}^n E_k)) = \sum_{k=1}^n \mu^*(A \cap E_k), \quad \forall\, n \in \mathbf{N}. \qquad (3.4)$$

Now $(3.1)'$, (3.4) and the monotonicity of μ^*, imply

$$\begin{aligned}\mu^*(A) &= \mu^*(A \cap (\cup_{k=1}^n E_k)) + \mu^*(A \cap (\cup_{k=1}^n E_k)^c)\\ &\geq \sum_{k=1}^n \mu^*(A \cap E_k) + \mu^*(A \cap (\cup_{k\in\mathbf{N}} E_k)^c)\,. \end{aligned} \tag{3.5}$$

We can take the limit in (3.5) as $n \to \infty$, and obtain from the countable subadditivity of μ^* that

$$\begin{aligned}\mu^*(A) &\geq \sum_{k\in\mathbf{N}} \mu^*(A \cap E_k) + \mu^*(A \cap (\cup_{k\in\mathbf{N}} E_k)^c)\\ &\geq \mu^*(A \cap (\cup_{k\in\mathbf{N}} E_k)) + \mu^*(A \cap (\cup_{k\in\mathbf{N}} E_k)^c). \end{aligned} \tag{3.6}$$

This shows that $\mathcal{M}$ is a σ-algebra.

We have proved (3.6) assuming $\mu^*(A) < \infty$, but of course this inequality holds also for $\mu^*(A) = \infty$. Taking $A = \cup_{k\in\mathbf{N}} E_k$ in (3.6) gives $\mu^*(\cup_{k\in\mathbf{N}} E_k) = \sum_{k\in\mathbf{N}} \mu^*(E_k)$, so that μ^* is countably additive on $\mathcal{M}$.

3.2 Theorem : Generation of Outer Measures. *Let $\mathcal{E}$ be a family of subsets of Ω, which includes Ω as well as the empty set $\emptyset$. Given* any *set-function $\nu : \mathcal{E} \to [0,\infty]$ satisfying $\nu(\emptyset) = 0$, the set-function*

$$\mu^*(A) := \inf\left\{ \sum_{n\in\mathbf{N}} \nu(E_n) \;\middle|\; \{E_n\}_{n\in\mathbf{N}} \subseteq \mathcal{E}\,,\; A \subseteq \cup_{n\in\mathbf{N}} E_n \right\}, \quad A \subseteq \Omega \tag{3.7}$$

defines an outer measure on Ω, the outer measure generated by ν.

Proof: By taking $E_n \equiv \emptyset$ in (3.7), we see that $\mu^*(\emptyset) = 0$. Also $\mu^*(A) \leq \mu^*(B)$ whenever $A \subseteq B$, since any covering of B is also a covering of A.

Finally, let $\{A_k\}_{k\in\mathbf{N}}$ be a sequence of subsets of Ω, and take any $\varepsilon > 0$. For each $k \in \mathbf{N}$, select a covering $\{E_{k,n}\}_{n\in\mathbf{N}} \subseteq \mathcal{E}$ of A_k, that is, $A_k \subseteq \cup_{n\in\mathbf{N}} E_{k,n}$, such that $\sum_{n\in\mathbf{N}} \nu(E_{k,n}) \leq \mu^*(A_k) + \varepsilon 2^{-k}$.

Then $\cup_{k\in\mathbf{N}} A_k \subseteq \cup_{k\in\mathbf{N}} \cup_{n\in\mathbf{N}} E_{k,n}$ and $\mu^*(\cup_{k\in\mathbf{N}} A_k) \leq \sum_{k\in\mathbf{N}} \sum_{n\in\mathbf{N}} \nu(E_{k,n}) \leq \sum_{k\in\mathbf{N}} \mu^*(A_k) + \varepsilon$. Since ε was arbitrary, we conclude that μ^* is countably subadditive. $\diamond$

Although Theorem 3.2 associates an outer measure μ^* to *any* given set-function ν that satisfies its conditions, the outer measure μ^* may be quite different from ν, even when restricted to sets in the original family $\mathcal{E}$. The following theorem gives conditions, under which ν can be extended to a measure on $\sigma(\mathcal{E})$, and in a unique manner (cf. Exercise 3.9). We shall need a couple of new notions.

Let us recall that a family $\mathcal{E}$ of subsets of Ω is called an *algebra*, if it contains the empty set and is closed under complements and under finite unions. It is not hard to see that an algebra is a σ-algebra, if it is closed under countable *disjoint* unions.

3.2 Definition: Pre-Measure on an Algebra. Given an algebra $\mathcal{E}$ of subsets of Ω, a set-function $\nu : \mathcal{E} \to [0, \infty]$ is called *pre-measure*, if it satisfies $\nu(\emptyset) = 0$ and $\nu(\cup_{n\in\mathbf{N}} E_n) = \sum_{n\in\mathbf{N}} \nu(E_n)$ for any sequence $\{E_n\}_{n\in\mathbf{N}} \subseteq \mathcal{E}$ of pairwise disjoint sets with $\cup_{n\in\mathbf{N}} E_n \in \mathcal{E}$.

3.3 THEOREM: HAHN EXTENSION. *Suppose that the family $\mathcal{E}$ of Theorem 3.2 is an algebra, and that the set-function $\nu : \mathcal{E} \to [0, \infty]$ is a* pre-measure *on $\mathcal{E}$.*

Denote by μ^* the outer measure generated by ν as in (3.7), and by $\mu := \mu^*\big|_{\sigma(\mathcal{E})}$ the restriction of this outer measure to the σ-algebra $\sigma(\mathcal{E})$ generated by $\mathcal{E}$.

(i) *Then $\sigma(\mathcal{E}) \subseteq \mathcal{M}$, where $\mathcal{M}$ is the σ-algebra of Theorem 3.1, and $\mu\big|_{\mathcal{E}} = \mu^*\big|_{\mathcal{E}} = \nu$.*

(ii) *For any measure ρ on $\sigma(\mathcal{E})$ which satisfies $\rho\big|_{\mathcal{E}} = \nu$, we have*

$$\rho(A) \le \mu(A)\,, \quad \forall\ A \in \sigma(\mathcal{E}) \qquad (\textit{with equality when } \mu(A) < \infty).$$

If ν is σ-finite, *then μ is the* unique *measure on $\sigma(\mathcal{E})$ with $\mu\big|_{\mathcal{E}} = \nu$; to wit, the unique extension of the pre-measure ν on $\mathcal{E}$ to a measure on $\sigma(\mathcal{E})$.*

Proof : Since $\mathcal{M}$ is a σ-algebra, to prove $\sigma(\mathcal{E}) \subseteq \mathcal{M}$ it is enough to show $\mathcal{E} \subseteq \mathcal{M}$, i.e., that every set $E \in \mathcal{E}$ satisfies (3.1) for any $A \subseteq \Omega$. To this end, fix $\varepsilon > 0$ and find a sequence $\{Z_k\}_{k\in\mathbf{N}} \subseteq \mathcal{E}$ with $A \subseteq \cup_{k\in\mathbf{N}} Z_k$ and

$$\mu^*(A) + \varepsilon \ \ge\ \sum_{k\in\mathbf{N}} \nu(Z_k) \ =\ \sum_{k\in\mathbf{N}} [\nu(Z_k \cap E) + \nu(Z_k \cap E^c)] \ \ge\ \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Here we made use of the definition of outer measure, of the fact that $Z_k \cap E$ and $Z_k \cap E^c$ belong to $\mathcal{E}$, and of the additivity of ν on $\mathcal{E}$. Since $\varepsilon > 0$ was arbitrary, it follows $E \in \mathcal{M}$.

- We show next that $\mu^*(A) = \nu(A)$ holds for every $A \in \mathcal{E}$. Let $\{E_k\}_{k\in\mathbf{N}} \subseteq \mathcal{E}$ be any covering of A, namely $A \subseteq \cup_{k\in\mathbf{N}} E_k$, with sets in $\mathcal{E}$. Then $B_n = A \cap (E_n \setminus \cup_{k=1}^{n-1} E_k)$, $n \in \mathbf{N}$ are disjoint sets in $\mathcal{E}$, with $\cup_{n\in\mathbf{N}} E_n \supseteq \cup_{n\in\mathbf{N}} B_n = A \in \mathcal{E}$. Thus

$$\nu(A) \ =\ \sum_{n\in\mathbf{N}} \nu(B_n) \ \le\ \sum_{n\in\mathbf{N}} \nu(E_n)$$

and this gives $\nu(A) \le \mu^*(A)$. The reverse inequality $\mu^*(A) \le \nu(A)$ is obvious, since we can construct a covering of A in (3.7) by taking $E_1 = A$, $E_n = \emptyset$ for $n \ge 2$. *Part (i)* of the theorem is proved.

- To establish *Part (ii),* let $A \in \sigma(\mathcal{E})$ and suppose that

$$\{E_k\}_{k\in\mathbf{N}} \subseteq \mathcal{E} \quad \text{is } \textit{any} \text{ sequence with } A \subseteq \cup_{k\in\mathbf{N}} E_k =: E \in \mathcal{E}\,. \qquad (3.8)$$

Then $\rho(A) \le \rho(E) \le \sum_{k\in\mathbf{N}} \rho(E_k) = \sum_{k\in\mathbf{N}} \nu(E_k)$; because the sequence $\{E_k\}_{k\in\mathbf{N}} \subseteq \mathcal{E}$ is arbitrary and $A \in \sigma(\mathcal{E})$, $\mu = \mu^*\big|_{\sigma(\mathcal{E})}$, we deduce $\rho(A) \le \mu^*(A) = \mu(A)$.

If $\mu(A) = \mu^*(A) < \infty$, then given any $\varepsilon > 0$ we can choose the sets in the sequence $\{E_k\}_{k\in\mathbf{N}}$ of (3.8) to be pairwise disjoint and to satisfy

$$\mu(A) \le \mu(E) \le \sum_{k\in\mathbf{N}} \mu(E_k) = \sum_{k\in\mathbf{N}} \rho(E_k) = \sum_{k\in\mathbf{N}} \nu(E_k) \le \mu^*(A) + \varepsilon = \mu(A) + \varepsilon\,,$$

and in particular $\mu(E \setminus A) \le \varepsilon$. Recallling that both A, E (thus also $E \setminus A$) are in $\sigma(\mathcal{E})$, we deduce

$$\mu(A) \le \mu(E) \le \sum_{k\in\mathbf{N}} \rho(E_k) = \rho(E) = \rho(A) + \rho(E \setminus A) \le \rho(A) + \mu(E \setminus A) \le \rho(A) + \varepsilon.$$

Since this is true for any $\varepsilon > 0$, we obtain $\mu(A) \le \rho(A)$.

The last statement of the theorem also follows easily: in the case of σ-finite ν we have $\Omega = \cup_{k\in\mathbf{N}}\Omega_k$, where the sets $\{\Omega_k\}_{k\in\mathbf{N}} \subseteq \mathcal{E}$ are pairwise disjoint with $\mu(\Omega_k) = \nu(\Omega_k) < \infty$ for every $k \in \mathbf{N}$, and thus $\mu(A) = \sum_{k\in\mathbf{N}} \mu(A \cap \Omega_k) = \sum_{k\in\mathbf{N}} \rho(A \cap \Omega_k) = \rho(A)$ for any $A \in \sigma(\mathcal{E})$. The proof is complete.

3.3 Definition : Elementary Family of Sets. A collection $\mathcal{G}$ of subsets of Ω is called *elementary family,* if it contains the empty set, is closed under finite intersections, and the complement of every $E \in \mathcal{G}$ can be written as a finite union $E^c = \cup_{j=1}^n F_j$ of pairwise-disjoint sets $\{F_j\}_{j=1}^n \subseteq \mathcal{G}$.

For instance, the collection consisting of all half-open intervals $(a, b]$ with $-\infty < a \le b \le \infty$ of the real line, is an elementary family. (When $b = \infty$, we interpret the interval as the half-line (a, ∞); when $a = b$, as the empty set).

3.1 Exercise : Let $\mathcal{G}$ be an elementary family, and consider the class $\mathcal{E}$ of finite disjoint unions of sets in $\mathcal{G}$. Show that $\mathcal{E}$ is an algebra.

3.2 Exercise : An outer measure μ^* is a measure, if $\mu^*\left(\cup_{j=1}^n A_j\right) = \sum_{j=1}^n \mu^*(A_j)$ holds for any finite collection $A_1, \cdots, A_n$ of disjoint sets.

3.3 Exercise : Suppose that μ is a measure on the real line, finite on bounded Borel sets. Then $F(x) = \mu((0, x])$, $x > 0$, $F(0) = 0$, $F(x) = -\mu((x, 0])$, $x < 0$ defines an increasing, right-continuous function. We also have the analogue $\mu((a, b]) = F(b) - F(a)$ of (1.7).

A: COMPLETENESS OF MEASURE SPACES

Occasionally we want to show that some given function g is $\mathcal{F}-$measurable. We identify a function f which we know has this property, and then manage to show that $\{f \neq g\}$ is (contained in) a set of zero measure. Can we conclude that g is $\mathcal{F}-$measurable from this? The answer is *yes, if the measure space has the property of* **completeness**; see Definition 3.5 below. We introduce this very important property in this subsection, and study its ramifications in a series of exercises.

3.4 Definition : Null Sets. A measurable set E is called *null* for the measure μ, if $\mu(E) = 0$. By subadditivity of μ, any countable union of null sets is again null.

The union of an uncountable collection of null sets can easily fail to be a null set.

3.5 Definition : Negligible Sets and Completeness. We say that a subset $F \subseteq \Omega$ in a measure space $(\Omega, \mathcal{F}, \mu)$ is *μ-negligible*, and write $F \in \mathcal{N}$, if there exists a null set E with $F \subseteq E$. If $\mathcal{N} \subseteq \mathcal{F}$, we say that the measure(-space) is *complete.*

3.6 Definition : Almost-everywhere and modulo μ. A given statement is said to hold *$\mu-$almost everywhere (a.e.),* if the set on which it fails is negligible.

Two classes $\mathcal{A}$, $\mathcal{B}$ of subsets of Ω are said to *agree modulo* μ, and we write $\mathcal{A} = \mathcal{B}$ modulo μ, if $\mathcal{A} \setminus \mathcal{B} \subseteq \mathcal{N}$ and $\mathcal{B} \setminus \mathcal{A} \subseteq \mathcal{N}$.

Completeness is a very useful property for a measure-space (see Exercise 3.6, for examples) and simplifies many technical arguments. It can always be achieved by suitably enlarging the domain of μ, as the following exercises demonstrate.

3.4 Exercise : Completion of a Measure Space. (i) In the context of Definition 3.4, show that $\overline{\mathcal{F}} := \{E \cup F \,|\, E \in \mathcal{F},\, F \in \mathcal{N}\}$ is a σ-algebra, that $\overline{\mu}(E \cup F) := \mu(E)$ defines well a complete measure on $\overline{\mathcal{F}}$, and that this $\overline{\mu}$ is the *unique* extension of μ to a complete measure on $\overline{\mathcal{F}}$. The measure space $(\Omega, \overline{\mathcal{F}}, \overline{\mu})$ is called the *completion of* $(\Omega, \mathcal{F}, \mu)$; these two spaces coincide, if the original measure-space is complete.
(ii) If $f : \Omega \to \mathbf{R}$ is $\overline{\mathcal{F}}-$measurable, then there exists an $\mathcal{F}-$measurable function $g : \Omega \to \mathbf{R}$ such that $f = g$, $\overline{\mu}-$a.e.

3.5 Exercise : Completeness in the Carathéodory construction.
(i) Show that the measure $\overline{\mu} := \mu^*\big|_{\mathcal{M}}$ of Theorem 3.1 is complete.
(ii) If μ^* is the outer-measure generated by a pre-measure ν on an algebra $\mathcal{E}$, in the manner of Theorems 3.1 and 3.2, then this measure's restriction $\mu^*\big|_{\sigma(\mathcal{E})} =: \mu$ to $\sigma(\mathcal{E})$ need not be complete.
(iii) Show, however, that the completion of the measure space $(\Omega, \sigma(\mathcal{E}), \mu)$ is the measure space $(\Omega, \mathcal{M}, \overline{\mu})$ where $\overline{\mu} := \mu^*\big|_{\mathcal{M}}$ and μ^* is the outer measure of part (i), provided this outer measure is σ-finite.

(*Hint:* Every $A \subseteq \Omega$ has a "measurable cover", that is, a set $B \in \mathcal{F}$ with $A \subseteq B$, $\mu^*(A) = \mu^*(B)$. And every $E \in \mathcal{M}$ can be written as $E = B \cup N$ with $B \in \mathcal{F}$, $N \subseteq D$ for some $D \in \mathcal{F}$ with $\mu^*(D) = 0$. Now follow Exercise 3.2, Theorem 3.1.)

3.6 Exercise: The benefits of Completeness. Let $(\Omega, \mathcal{F}, \mu)$ be a *complete* measure space, and consider real-valued functions f, g, $\{f_n\}_{n \in \mathbf{N}}$ on it.

(i) If f is measurable and $f = g$ holds $\mu-$a.e., then g is also measurable.

(ii) If $\{f_n\}_{n \in \mathbf{N}}$ are measurable and $\lim_n f_n = f$ holds $\mu-$a.e., then f is also measurable.

B: MONOTONE CLASSES AND DYNKIN SYSTEMS

In measure theory we face often the following situation: We want to show that a certain property holds for all sets in a certain $\sigma-$algebra $\mathcal{G}$, but we can do this relatively easily only for sets in a subclass $\mathcal{D}$ of $\mathcal{G}$. Under what conditions on $\mathcal{G}$ and $\mathcal{D}$ can we then ensure that the property holds on the larger class $\mathcal{G}$?

The following two exercises introduce concepts and results that make such conclusions possible, in a fairly systematic way. They will be used rather extensively in what follows, so the reader will be well advised to think them through very carefully.

3.7 EXERCISE: Monotone Class Theorem. A non-empty collection $\mathcal{M}$ of subsets of a non-empty space Ω is called a *Monotone Class,* if it is closed under countable increasing unions and countable decreasing intersections. Clearly, every σ-algebra is a monotone class.

(i) The intersection of an arbitrary family of monotone classes is a monotone class; thus, for any family $\mathcal{E}$ of subsets of Ω, there is a smallest monotone class – denoted by $m(\mathcal{E})$ – that contains $\mathcal{E}$.

(ii) If $\mathcal{E}$ is an algebra, then $\sigma(\mathcal{E}) = m(\mathcal{E})$.

3.8 EXERCISE : Dynkin System Theorem. A non-empty collection $\mathcal{D}$ of subsets of a non-empty space Ω is called a

- $\pi-$*system,* if it is closed under finite intersections; it is called a

- $\lambda-$*system,* if it contains Ω, is closed under countable increasing unions, and $A \setminus B \in \mathcal{D}$ whenever $A \in \mathcal{D}$, $B \in \mathcal{D}$ and $B \subseteq A$.

The following hold:

(i) If $\mathcal{D}$ is both a $\pi-$system and a $\lambda-$system, then it is a σ-algebra (and vice-versa).

(ii) If a $\lambda-$system $\mathcal{A}$ contains a $\pi-$system $\mathcal{D}$, then it contains also the σ-algebra $\sigma(\mathcal{D})$ generated by $\mathcal{D}$.

(iii) Suppose $\mathcal{D}$ is a $\pi-$system, and we have two measures μ and ν on $(\Omega, \sigma(\mathcal{D}))$ with $\mu(\Omega) = \nu(\Omega) < \infty$ and $\mu \equiv \nu$ on $\mathcal{D}$; then $\mu \equiv \nu$ on $\sigma(\mathcal{D})$.

In particular, if two probability measures agree on a $\pi-$system $\mathcal{D}$, then they agree also on the σ-algebra $\sigma(\mathcal{D})$ generated by the $\pi-$system.

3.9 Exercise : The collection $\mathcal{E}$ of finite unions of sets $(a, b] \cap \mathbf{Q}$ with $-\infty \leq a \leq b \leq \infty$ is an algebra on $\mathbf{Q}$, the rational real numbers, and $\sigma(\mathcal{E}) = \mathcal{P}(\mathbf{Q})$.

If we define $\nu(\emptyset) = 0$ and $\nu(A) = \infty$ for $A \neq \emptyset$, then ν is a pre-measure on $\mathcal{E}$; and there is more than one measure μ on $\mathcal{P}(\mathbf{Q})$ with $\mu|_{\mathcal{E}} = \nu$.