1.5. INEQUALITIES AND FUNCTION SPACES

For any given real number $p > 0$, let us denote by $\mathbf{L}^p \equiv \mathbf{L}^p(\mu)$ the set of measurable, real-valued functions f on the measure-space $(\Omega, \mathcal{F}, \mu)$ with $|f|^p \in \mathbf{L}^1$, or equivalently

$$||f||_p := \left(\int_\Omega |f|^p \, d\mu \right)^{1/p} < \infty \, . \tag{5.1}$$

The notation is in honor of H. Lebesgue.

The elementary properties $|f+g|^p \le (2 \cdot \max(|f|, |g|))^p \le 2^p \cdot (|f|^p + |g|^p)$ and $||\alpha f||_p = |\alpha| \, ||f||_p$ for $\alpha \in \mathbf{R}$, show that $\mathbf{L}^p$ is a real vector space. We shall say that a sequence $\{f_n\}_{n \in \mathbf{N}}$ of functions in $\mathbf{L}^p$ *converges to some function* f *in* $\mathbf{L}^p$, if $\lim_{n \to \infty} ||f_n - f||_p = 0$.

On the other hand, the "triangle inequality" (5.3) below shows that, for $1 \le p < \infty$, the quantity $|| \cdot ||_p$ of (5.1) is a norm on $\mathbf{L}^p$.

Important Remark: We employ here and in the sequel the "usual convention" of identifying functions that are equal $\mu-$a.e. on Ω; for instance, we identify $f = \chi_{\mathbf{Q}}$ with $g = 0$ on the real line with Lebesgue measure. Thus, we are (tacitly) treating $\mathbf{L}^p$ as a space of *equivalence classes* of functions, rather than as a space of functions.

HÖLDER INEQUALITY: *For any* $p \in (1, \infty)$*, define* q *by* $(1/p) + (1/q) = 1$; *then for any measurable, real-valued functions* f, g *we have*

$$||fg||_1 \le ||f||_p \cdot ||g||_q \, . \tag{5.2}$$

If $f \in \mathbf{L}^p$ *and* $g \in \mathbf{L}^q$, *this shows* $fg \in \mathbf{L}^1$, *and in this case (5.2) holds as equality* if and only if *there exist real constants* α, β *such that* $\alpha \beta \neq 0$ *and* $\alpha |f|^p = \beta |g|^q$, $\mu-$a.e.

For $p = 2$, the inequality (5.2) is known as the **Cauchy-Schwarz inequality.**

MINKOWSKI INEQUALITY: *For any* $p \in [1, \infty)$, *we have the* triangle inequality

$$||f+g||_p \le ||f||_p + ||g||_p \, , \quad \forall \;\; f, g \; \in \mathbf{L}^p. \tag{5.3}$$

5.1 Exercise: *The triangle inequality (5.3) fails for* $p \in (0,1)$. (*Hint:* Justify the elementary inequality $(a+b)^p < a^p + b^p$ for $a > 0$, $b > 0$, $0 < p < 1$, and write it with $a = (\mu(E))^{1/p}$, $b = (\mu(F))^{1/p}$ for any two disjoint measurable sets E, F of positive measure.)

5.2 Exercise: For $a \ge 0$, $b \ge 0$, $0 < \lambda < 1$ we have $a^\lambda \cdot b^{1-\lambda} \le \lambda a + (1-\lambda) b$, with equality iff $a = b$. (*Hint:* The function $\xi(u) = u^\lambda - u\lambda$ attains its maximum, namely $1 - \lambda$, over the half-line $[0, \infty)$, at $u = 1$.)

Proof of (5.2) : The inequality is obvious when its right-hand side $||f||_p \cdot ||g||_q$ vanishes, so let us assume $||f||_p > 0,\ ||g||_q > 0$. Then we can read the inequality of Exercise 5.2 with $a = (|f(\omega)|/||f||_p)^p$, $b = (|g(\omega)|/||g||_q)^q$, $\lambda = (1/p)$, to wit

$$\frac{|f(\omega)\, g(\omega)|}{||f||_p\, ||g||_q} \le \frac{|f(\omega)|^p}{p\, I(|f|^p)} + \frac{|g(\omega)|^q}{q\, I(|g|^q)}, \qquad \text{for } \omega \in \Omega$$

(with equality, iff $I(|g|^q) \cdot |f(\omega)|^p = I(|f|^p) \cdot |g(\omega)|^q$ holds for $\mu-$a.e. $\omega \in \Omega$) ; integrating over Ω with respect to μ, we obtain $(||fg||_1) \,/\, (||f||_p\, ||g||_q\,) \le (1/p) + (1/q) = 1$.

Proof of (5.3) : The inequality is quite clear for $p = 1$, as well as when $f + g = 0$, $\mu-$a.e. Now for $p > 1$ and $\mu(f + g \neq 0) > 0$, we start by writing $|f + g|^p \le |f| \cdot |f + g|^{p-1} + |g| \cdot |f + g|^{p-1}$; integrating with respect to μ and then applying Hölder's inequality to the right-hand side, we obtain

$$\begin{aligned} I\left(|f+g|^p\right) &\le ||f||_p \cdot ||(|f+g|)^{p-1}||_q + ||g||_p \cdot ||(|f+g|)^{p-1}||_q \\ &\le (||f||_p + ||g||_p) \cdot \left(I\left(|f+g|^{q(p-1)} \right) \right)^{1/q} = (||f||_p + ||g||_p) \cdot \left(I\left(|f+g|^p\right) \right)^{1/q} \end{aligned}$$

whence $\left(I\left(|f+g|^p\right)\right)^{1-(1/q)} = ||f+g||_p \le ||f||_p + ||g||_p$. ◇

Let us recall from Appendix C that a function $F : (a, b) \to \mathbf{R}$ is called *convex*, if the property $F\left(\sum_{k=1}^K \lambda_k y_k\right) \le \sum_{k=1}^K F\left(\lambda_k y_k\right)$ holds for any $y_1, \dots y_K$ in (a, b) and any $\lambda_1, \dots, \lambda_K$ in $[0, 1]$ with $\lambda_1 + \dots + \lambda_K = 1$ and any $K \in \mathbf{N}$. In particular, if we interpret $\{y_1, \dots y_K\}$ as the range of a simple function h on a probability space $(\Omega, \mathcal{F}, \mu)$, and λ_k as $\mu\big(h^{-1}(\{y_k\})\big)$, then the inequality reads $F\big(I(h)\big) \le I\big(F(h)\big)$, and is actually valid for any integrable function h, as the following result demonstrates.

JENSEN INEQUALITY: *Suppose that μ is a probability measure, that $h : \Omega \to (a, b)$ is in $\mathbf{L}^1(\mu)$, and that $F : (a, b) \to \mathbf{R}$ is a convex function, for some $-\infty \le a < b \le \infty$. We have then*

$$F\big(I(h)\big) \le I\big(F(h)\big). \tag{5.4}$$

Proof : Since $F(\cdot)$ is convex, its left- and right-derivatives $D^- F(\cdot) \le D^+ F(\cdot)$ exist everywhere on (a, b), are nondecreasing functions, and $F(\cdot) = F(x_0) + \int_{x_0}^{\cdot} D^\pm F(u)\, du$ for fixed $x_0 \in (a, b)$; in particular, $F(\cdot)$ is continuous (Properties $C.1 - C.3$ in Appendix C; Exercise 4.4). Now for any given $s \in (a, b)$, there exists a real number β such that

$$F(t) - F(s) \ \ge\ \beta(t - s)\,, \qquad \forall\ t \in (a, b)\,;$$

just take $\beta \in [D^- F(s), D^+ F(s)]$. With $s = I(h)$, $t = h(\omega)$ this inequality reads: $F(h(\omega)) - F(I(h)) \ge \beta \cdot (h(\omega) - I(h))$, $\forall\ \omega \in \Omega$. Thus $F(h)$ is measurable (as the composition of the measurable h with the continuous F) and its integral is well-defined (as $F(h)$ is bounded from below by an integrable function). Integrate over Ω with respect to μ and recall $\mu(\Omega)=1$, to obtain $I\big(F(h)\big)-F\big(I(h)\big)\cdot\mu(\Omega)\geq\beta\big(I(h)-I(h)\cdot\mu(\Omega)\big)=0$, that is, (5.4). $\diamond$

We can also define a space $\mathbf{L}^\infty\equiv\mathbf{L}^\infty(\mu)$, as the set of all measurable functions $f:\Omega\to\mathbf{R}$ which are essentially bounded, in the sense that the *essential least-upper-bound*

$$
\begin{aligned}
||f||_\infty &:= \inf\{\,a\geq 0\,:\,|f(\omega)|\leq a \text{ for a.e. } \omega\in\Omega\,\} \\
&= \sup\{a\geq 0\,|\,\mu(\{\omega\in\Omega\,:\,|f(\omega)|>a\})>0\}
\end{aligned}
\tag{5.5}
$$

is finite: $||f||_\infty<\infty$. Under the usual convention, it is straightforward to check that $\mathbf{L}^\infty$ is a real vector space with $||\cdot||_\infty$ as its norm.

The essential least-upper-bound ignores sets of measure zero; for instance, if $f(\omega)=1$ for rational $\omega\in\mathbf{R}$ and $f(\omega)=0$ otherwise, then $||f||_\infty=0$ but $\sup_{\omega\in\Omega}|f(\omega)|=1$.

A real-valued function f can easily have $||f||_\infty=\infty$. Just consider $(\mathbf{R},\mathcal{B}(\mathbf{R}))$ with the *Gaussian measure* $\mu(A)=(\sqrt{2\pi}\,)^{-1}\int_A e^{-x^2/2}dx$, and $f(\omega)=\omega,\ \omega\in\mathbf{R}$; note that $\mu(|f|>a)>0$ holds for every $a\geq 0$.

5.3 Exercise: (i) Note that the infimum in (5.5) is actually attained.

(ii) Suppose $f\in\mathbf{L}^\infty$. Then $|f(\omega)|\leq||f||_\infty$ for $\mu-$a.e. $\omega\in\Omega$; and for every $0<a<||f||_\infty$ there exists a set $E\in\mathcal{F}$ with $\mu(E)>0$ such that $|f(\omega)|>a$, $\forall\,\omega\in E$.

(iii) For any $f\in\mathbf{L}^1$, $g\in\mathbf{L}^\infty$, $h\in\mathbf{L}^\infty$ and $\{g_n\}_{n\in\mathbf{N}}\subseteq\mathbf{L}^\infty$, we have the analogue of the Hölder inequality $||fg||_1\leq||f||_1\cdot||g||_\infty$, the analogue of the Minkowski inequality $||g+h||_\infty\leq||g||_\infty+||h||_\infty$. We also have the equivalence

$$\lim_{n\to\infty}||g_n-g||_\infty=0 \quad\Longleftrightarrow\quad \lim_{n\to\infty}\left(\sup_{\omega\in E}|g_n(\omega)-g(\omega)|\right)=0$$

for some measurable set E with $\mu(E^c)=0$. In other words, convergence in $\mathbf{L}^\infty$ is uniform convergence outside a set of measure zero.

5.4 Exercise : Egorov's Theorem. Suppose g, $\{g_n\}_{n\in\mathbf{N}}$ are measurable functions on a *finite* measure space ($\mu(\Omega)<\infty$), and $\lim_{n\to\infty}g_n=g$ holds $\mu-$a.e. Then

(i) for every $\delta>0$, there exists a measurable set E with $\mu(E^c)<\delta$ and

$$\lim_{n\to\infty}\left(\sup_{\omega\in E}|g_n(\omega)-g(\omega)|\right)=0\,;$$

(ii) $\lim_{n\to\infty}\mu(|g_n-g|>\varepsilon)=0$, $\forall\,\varepsilon>0$;

(iii) the assumption $\mu(\Omega) < \infty$ can be replaced by "$|g_n| \le f$, $\forall n \in \mathbf{N}$ for some $f \in \mathbf{L}^1(\mu)$" in (i) above.

5.5 EXERCISE : CONVERGENCE IN MEASURE. Suppose f, g, $\{f_n\}_{n\in\mathbf{N}}$, $\{g_n\}_{n\in\mathbf{N}}$ are elements of $\mathbf{L}^0$, the space of measurable, real-valued functions on a complete measure space $(\Omega, \mathcal{F}, \mu)$. We say that the sequence $\{g_n\}_{n\in\mathbf{N}}$ **converges in measure** to g, if

$$\lim_{n\to\infty} \mu(|g_n - g| > \varepsilon) = 0 , \quad \forall\, \varepsilon > 0 .$$

(i) Argue that the limit g is unique modulo $\mu-$a.e. equality. Also, observe that *$\mu-$a.e. convergence implies convergence in measure, if $\mu(\Omega) < \infty$.*

(ii) If $\mu(\Omega) < \infty$, show that

$$\rho(f,g) \equiv I\left(\frac{|f-g|}{1+|f-g|}\right)$$

defines a *metric* on the space $\mathbf{L}^0$ of measurable, real-valued functions, and that convergence in this metric is equivalent to convergence in measure. Under this metric $\mathbf{L}^0$ becomes a *Fréchet space* (complete, metrizable vector space).

Similarly, with $\varrho(f,g) := I(|f-g| \wedge 1)$.

(iii) Show that convergence in $\mathbf{L}^p$, for some $p > 0$, implies convergence in measure. (*Hint:* Recall the Čebyšev inequality of (2.14).)

(iv) Show by example, that $\mu-$a.e. convergence does not imply convergence in $\mathbf{L}^p$; that $\mu-$a.e. convergence does not imply convergence in measure if $\mu(\Omega) = \infty$; and that convergence in $\mathbf{L}^p$ does not imply $\mu-$a.e. convergence.

(v) Suppose that the sequence $\{g_n\}_{n\in\mathbf{N}}$ is *"Cauchy in measure"*, i.e.,

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \mu(|g_n - g_m| > \varepsilon) = 0 , \quad \forall\, \varepsilon > 0 .$$

Then there exists a measurable function $g : \Omega \to \mathbf{R}$ such that $\{g_n\}_{n\in\mathbf{N}}$ converges in measure to g; as well as a subsequence $\{g_{n_k}\}_{k\in\mathbf{N}}$ which converges to g, $\mu-$a.e.

(vi) Suppose $\{f_n\}_{n\in\mathbf{N}}$ (resp., $\{g_n\}_{n\in\mathbf{N}}$) converges in measure to f (resp., g). Show that:

- $\{f_n + g_n\}_{n\in\mathbf{N}}$ converges in measure to $f+g$;
- $\{f_n\, g_n\}_{n\in\mathbf{N}}$ converges in measure to $f\, g$ if $\mu(\Omega) < \infty$.
- $\{\varphi(f_n)\}_{n\in\mathbf{N}}$ converges in measure to $\varphi(f)$, provided $\varphi : \mathbf{R} \to \mathbf{R}$ is uniformly continuous;
- $\{\varphi(f_n)\}_{n\in\mathbf{N}}$ converges in measure to $\varphi(f)$, provided $\varphi : \mathbf{R} \to \mathbf{R}$ is continuous and $\mu(\Omega) < \infty$. More generally, $\{\Phi(f_n, g_n)\}_{n\in\mathbf{N}}$ converges in measure to $\Phi(f,g)$, for any continuous function $\Phi : \mathbf{R}^2 \to \mathbf{R}$ and $\mu(\Omega) < \infty$.

- *Fatou's Lemma:* $I(f) \le \liminf_{n\to\infty} I(f_n)$, provided $f_n \ge 0, \ \forall\, n \in \mathbf{N}$; and
- *Dominated Convergence Theorem:* $f \in \mathbf{L}^1(\mu)$, $\lim_{n\to\infty} I(|f_n - f|) = 0$ and $I(f) = \lim_{n\to\infty} I(f_n)$, provided that $|f_n| \le h, \ \forall\, n \in \mathbf{N}$ holds for some $h \in \mathbf{L}^1(\mu)$.

5.6 Exercise: If $0 < p < q < r \le \infty$, then $\mathbf{L}^p \cap \mathbf{L}^r \subseteq \mathbf{L}^q$; in fact,

$$||f||_q \le (\,||f||_p)^{\ell}\,\,(\,||f||_r)^{1-\ell}\,, \qquad \text{with } \ \ell \in [0,1] \ \text{ defined via } \quad \frac{1}{q} = \frac{\ell}{p} + \frac{1-\ell}{r}.$$

5.7 Exercise: Lyapunov Inequality. If $\mu(\Omega) < \infty$ and $0 < p < q \le \infty$, then $\mathbf{L}^q \subseteq \mathbf{L}^p$; in particular

$$||f||_p \le ||f||_q \cdot (\mu(\Omega))^r\,, \qquad \text{with} \qquad r = (1/p) - (1/q)\,;$$

and when μ is a probability measure, this leads to the *Lyapunov Inequality*

$$||f||_p \le ||f||_q\,.$$

5.8 Exercise: *If* $\mu(\Omega) = \infty$, *the conclusions of the previous Exercise do not hold;* in fact, $\mathbf{L}^p$ may then fail to be a subset of $\mathbf{L}^q$, *for all* $p \neq q$.

(i) Show by example that, when $\mu(\Omega) = \infty$, we may have $\{p \in [1,\infty)\,|\, f \in \mathbf{L}^p\} = (r,s)$, a proper subinterval of $(1,\infty)$.

(ii) Take $\Omega = (0,\infty)$ with Lebesgue measure, and show that $f(\omega) = \frac{1}{\sqrt{\omega}\,(1+|\log \omega|)}$ is in $\mathbf{L}^p$ only for $p = 2$.

5.9 Exercise: Differentiating under the integral. Let $[a,b]$ be a given bounded interval of the real line, let $f : [a,b] \times \Omega \to \mathbf{R}$ a measurable function such that $f(t,\cdot)$ is in $\mathbf{L}^1(\mu)$ for every $t \in [a,b]$, and define the function $F(t) = \int_\Omega f(t,\omega)\, d\mu(\omega)$, $t \in [a,b]$.

(i) Suppose that there is a $g \in \mathbf{L}^1(\mu)$ such that $|f(t,\omega)| \le g(\omega), \ \forall\,(t,\omega) \in [a,b] \times \Omega$. If $\lim_{s\to t} f(s,\omega) = f(t,\omega), \ \forall\, \omega \in \Omega$, then $\lim_{s \to t} F(s) = F(t)$. In particular, if $f(\cdot,\omega)$ is continuous for each $\omega \in \Omega$, then F is continuous.

(ii) Suppose that the partial derivative $\frac{\partial f}{\partial t}$ exists, and satisfies $|\frac{\partial f}{\partial t}(t,\omega)| \le h(\omega), \ \forall\,(t,\omega) \in [a,b] \times \Omega$ for some $h \in \mathbf{L}^1(\mu)$. Then F is differentiable, and *"we can differentiate under the integral sign":* $F'(t) = \int_\Omega \frac{\partial f}{\partial t}(t,\omega)\, d\mu(\omega)$.

5.10 EXERCISE : DUALITY OF $\mathbf{L}^p-$SPACES: (i) For any $p \in [1,\infty)$ and $f \in \mathbf{L}^p(\mu)$, the mapping $g \mapsto T_f(g) := I(fg)$ defines a bounded, linear operator on $\mathbf{L}^q(\mu)$ with $(1/p) + (1/q) = 1$, whose norm is

$$||T_f|| := \sup\left\{ \left| \int_\Omega fg\, d\mu \right| \ : \ g \in \mathbf{L}^q(\mu)\,, \, ||g||_q = 1 \right\} = ||f||_p < \infty\,. \qquad (5.6)$$

This holds also for $p = \infty$, if the measure μ is *semi-finite;* that is, if for every $E \in \mathcal{F}$ with $\mu(E) = \infty$, there exists $F \in \mathcal{F}$ with $F \subset E$, $0 < \mu(F) < \infty$.
(ii) Conversely, suppose that μ is semi-finite and $p \in [1, \infty]$, $(1/p) + (1/q) = 1$. If f is measurable and such that $fg \in \mathbf{L}^1(\mu)$ for every g in the space $\mathcal{S}_0$ of simple functions that vanish outside a set of finite measure, and if

$$N(f) := \sup \left\{ \left| \int_\Omega fg \, d\mu \right| \; : \; g \in \mathcal{S}_0, \, ||g||_q = 1 \right\} < \infty ,$$

then $f \in \mathbf{L}^p(\mu)$ and $N(f) = ||f||_p$.

This last result can be construed as a *converse Hölder inequality.*

A: COMPLETENESS OF $\mathbf{L}^p$ SPACES

The following result shows that the $\mathbf{L}^p$ spaces of this section are Banach (that is, *complete* normed linear) spaces, in the topologies induced by the norms of (5.1), (5.5) for $1 \le p \le \infty$.

5.1 THEOREM: The space $\mathbf{L}^p$ is complete, for any $1 \le p \le \infty$. *In other words: For any Cauchy sequence $\{f_n\}_{n \in \mathbf{N}}$ in $\mathbf{L}^p$, i.e., with the property that for every $\varepsilon > 0$ there is an integer N_ε so that*

$$||f_n - f_m||_p \le \varepsilon \quad \textit{holds for any} \;\; n \ge N_\varepsilon \,, \;\; m \ge N_\varepsilon \,, \tag{5.7}$$

there exists a unique $f \in \mathbf{L}^p$ such that $||f_n - f||_p \to 0$ as $n \to \infty$.

Furthermore, there exists a subsequence $\{f_{n_k}\}_{k \in \mathbf{N}} \subseteq \{f_n\}_{n \in \mathbf{N}}$, as well as a function $F : \Omega \to [0, \infty)$ in $\mathbf{L}^p$, such that for $\mu-$a.e. $\omega \in \Omega$ we have:

$$|f_{n_k}(\omega)| \le F(\omega) \,, \quad \forall \; k \in \mathbf{N} \qquad \textit{and} \qquad \lim_{k \to \infty} f_{n_k}(\omega) = f(\omega) \,.$$

The argument involves a couple of ideas that are often used to great advantage in Analysis and in Probability; see, for instance, Theorem 2.4.1 (iii), as well as the proof of the Strong Law of Large Numbers (Theorem 2.3.2). The first idea, is that

- subsequences that converge "sufficiently fast" in $\mathbf{L}^p$ must converge also $\mu-$a.e., and the second idea, is that
- it is enough to show $\mathbf{L}^p$ convergence for *some* subsequence.

Proof : To see how these ideas work in our present context, observe that the Cauchy property (5.7) allows us to choose a subsequence $\{f_{n_k}\}$ with $||f_{n_{k+1}} - f_{n_k}||_p \le 2^{-k}$, for all $k \in \mathbf{N}$. The sequence of positive functions $\{F_k\}_{k \in \mathbf{N}}$ defined by

$$F_1(\omega) := |f_{n_1}(\omega)| \,, \quad F_{k+1}(\omega) := |f_{n_1}(\omega)| + \sum_{j=1}^{k} |f_{n_{j+1}}(\omega) - f_{n_j}(\omega)| \quad \text{for } k \in \mathbf{N}$$

satisfies $||F_k||_p \le ||f_{n_1}||_p + \sum_{j=1}^{k-1} 2^{-j} \le ||f_{n_1}||_p + 1$ from the triangle inequality, and increases $\mu-$a.e. to a function F; then Fatou's Lemma guarantees that F is in $\mathbf{L}^p$, hence also $\mu-$a.e. finite: $I(F^p) \le \underline{\lim}_k I((F_k)^p) \le \big(1 + ||f_{n_1}||_p\big)^p < \infty$.

As a result, the sequence

$$f_{n_{k+1}}(\omega) = f_{n_1}(\omega) + \sum_{j=1}^{k} \left[f_{n_{j+1}}(\omega) - f_{n_j}(\omega) \right], \quad k \in \mathbf{N}$$

converges absolutely for $\mu-$a.e. $\omega \in \Omega$ to some real number

$$f(\omega) := \lim_{k\to\infty} f_{n_k}(\omega) = f_{n_1}(\omega) + \sum_{j\in\mathbf{N}} \left[f_{n_{j+1}}(\omega) - f_{n_j}(\omega) \right].$$

Because $|f_{n_k}(\omega)| \le F(\omega)$ and $F \in \mathbf{L}^p$, we deduce from the Dominated Convergence Theorem that $f \in \mathbf{L}^p$ and $||f_{n_k} - f||_p \longrightarrow 0$ as $k \to \infty$, since $|f_{n_k} - f| \le F + |f| \in \mathbf{L}^p$.

Now let us argue that the *entire sequence* $\{f_n\}_{n\in\mathbf{N}}$ *must converge in* $\mathbf{L}^p$ *to this function* $f \in \mathbf{L}^p$. For any $\varepsilon > 0$ we can choose $K_\varepsilon \in \mathbf{N}$ large enough, so that $||f_{n_k} - f||_p \le \varepsilon/2$ holds for all $k \ge K_\varepsilon$. On the other hand, from (5.7) we can choose $N_\varepsilon \in \mathbf{N}$ large enough, so that $||f_n - f_{n_k}||_p \le \varepsilon/2$ holds for all $n \ge N_\varepsilon$ and for all $k \ge K_\varepsilon$. The triangle inequality now implies

$$||f_n - f||_p \le ||f_n - f_{n_k}||_p + ||f_{n_k} - f||_p \le \varepsilon, \quad \forall\, n \ge N_\varepsilon,$$

which shows that the entire sequence $\{f_n\}_{n\in\mathbf{N}}$ converges in $\mathbf{L}^p$ to the function f. ◇

The Hölder inequality places the Banach spaces of this Theorem in a formal *duality*, with $\mathbf{L}^q$ the dual of the space $\mathbf{L}^p$ for $1 \le p \le \infty$ when $(1/p) + (1/q) = 1$. This duality is studied in detail in section 1.8; recall Exercise 5.10, and consult Theorem 8.1 and Remark 8.1 in this regard.

Clearly, the space $\mathbf{L}^2$ of square-integrable functions is *self-dual* in this sense; it is also a *Hilbert space* with inner product $\langle f, g \rangle = \int_\Omega fg\, d\mu$, as discussed in Appendix *B*.

5.11 Exercise: If $1 \le p < \infty$, the simple functions of the form $f = \sum_{n=1}^{N} \alpha_n \chi_{E_n}$ with $\alpha_n \in \mathbf{R}$ and $\mu(E_n) < \infty$ for $n = 1, \dots, N$, $N \in \mathbf{N}$ are dense in $\mathbf{L}^p$.

5.12 Exercise: For any $f : \mathbf{R} \to \mathbf{R}$ and any $x \in \mathbf{R}$, define the "shift" $f_x(\cdot) \equiv f(x + \cdot)$. Show that, if $f \in \mathbf{L}^p(\mathbf{R}) \equiv \mathbf{L}^p(\mathbf{R}, \mathcal{B}(\mathbf{R}), \lambda)$ for some $1 \le p < \infty$, and if f is not identically equal to a constant (a.e.), then $\lim_{x\to 0} ||f_x - f||_p = 0$. Observe also that the result fails for $p = \infty$, as it amounts then to the requirement that f agree a.e. with a uniformly continuous function. (*Hint:* Establish the result first for continuous functions with compact support; then approximate.)

5.13 Exercise: Justifying the notation $||f||_\infty$ for the essential least-upper-bound. Show that if $f \in \mathbf{L}^r \cap \mathbf{L}^\infty$ for some $1 \le r < \infty$, then $f \in \mathbf{L}^p$ for any $p \in [r, \infty]$ and we have $||f||_\infty = \lim_{p\to\infty} ||f||_p$.

5.14 Exercise: Differentiating the $\mathbf{L}^p$−norm. With $1 < p < \infty$ and $f, \; g$ in $\mathbf{L}^p$, show that the convex function $F(u) := \int_\Omega |f + ug|^p \, d\mu$ is differentiable at $u = 0$, with derivative

$$F'(0) \;=\; p \int_\Omega |f|^{p-2} \, f \, g \, d\mu \,.$$

(*Hint:* Use the convexity of $x \mapsto |x|^p$.)

5.15 Exercise: Hanner Inequalities. Suppose f, g are in $\mathbf{L}^p$. With $1 \le p \le 2$, show

$$\Big(||f+g||_p\Big)^p + \Big(||f-g||_p\Big)^p \;\ge\; \Big(||f||_p + ||g||_p\Big)^p + \Big|\,||f||_p - ||g||_p\Big|^p \,, \tag{5.8}$$

$$\Big(||f+g||_p + ||f-g||_p\Big)^p + \Big|\,||f+g||_p - ||f-g||_p\Big|^p \;\le\; 2^p \left[\Big(||f||_p\Big)^p + \Big(||g||_p\Big)^p\right] . \tag{5.9}$$

For $2 \le p < \infty$, show that the inequalities are reversed; in particular, for $p = 2$ the inequalities lead to the Parallelogram Identity $(B.7)$ of Appendix B. For $p = 1$, the inequality of (5.8) is just the triangle inequality.

5.16 Exercise: Projection on a closed, convex set. Let $\mathcal{G}$ be a closed, convex subset of the space $\mathbf{L}^p$ for $1 < p < \infty$: i.e., $ug + (1-u)h$ belongs to $\mathcal{G}$ for every $0 \le u \le 1$, $g \in \mathcal{G}$, $h \in \mathcal{G}$; and every Cauchy sequence $\{z_n\}_{n\in\mathbf{N}} \subset \mathcal{G}$ converges in $\mathbf{L}^p$ to some $z \in \mathcal{G}$. Then, for any $f \notin \big(\mathbf{L}^p \setminus \mathcal{G}\big)$, there exists an element $g_* \in \mathcal{G}$ such that

$$\delta \;:=\; \inf_{g\in\mathcal{G}} ||f-g||_p \;=\; ||f-g_*||_p \,.$$

Furthemore,

$$\int_\Omega (g-g_*) \, \big(f-g_*\big) \, |f-g_*|^{p-2} \, d\mu \;\le\; 0 \,, \qquad \forall \; g \in \mathcal{G} \,.$$

(*Hint:* This is a generalization of the Projection in Hilbert-space result, Theorem $B.2$ of Appendix B, which corresponds to $p = 2$ here. Try the case $1 < p \le 2$ first, by following the same reasoning as in the proof of that result and using Exercises 5.15, 5.14 in lieu of the Parallelogram Identity $(B.7)$.)

B: UNIFORM INTEGRABILITY *

This subsection introduces the important, but rather technical, notion of uniform integrability. The reader may wish to skip this section entirely on first reading, and return to it after some familiarity with the contents of sections 1.7, 1.9 or of Chapter 2 has been acquired. The results of this section will be used in a crucial way in the proof of the Pointwise Ergodic Theorem 9.2.

Let us begin by placing ourselves, throughout this subsection, on a *finite measure space* $(\Omega, \mathcal{F}, \mu)$, that is $\mu(\Omega) < \infty$, and observe that for any function $f \in \mathbf{L}^1$ we have $\int_{\{|f|>\lambda\}} |f| \, d\mu \to 0$ as $\lambda \to \infty$. The notion of uniform integrability is a generalization of this property.

5.1 Definition: We say that a family $\{f_\alpha\}_{\alpha \in A}$ of real-valued, measurable functions, is *uniformly integrable,* if for every $\varepsilon > 0$, there exists $\lambda_\varepsilon > 0$ such that:

$$\sup_{\alpha \in A} \int_{\{|f_\alpha|>\lambda\}} |f_\alpha| \, d\mu < \varepsilon, \quad \forall \, \lambda \geq \lambda_\varepsilon .$$

Uniform integrability implies boundedness in $\mathbf{L}^1$, as the following result demonstrates.

5.1 Proposition: *The family $\{f_\alpha\}_{\alpha \in A}$ is uniform integrable, if and only if both conditions below hold:*

(a) Boundedness in $\mathbf{L}^1$: $\sup_{\alpha \in A} I(|f_\alpha|) =: K < \infty$;

(b) Uniform absolute continuity: *for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that, for every $B \in \mathcal{F}$ with $\mu(B) < \delta_\varepsilon$, and all $\alpha \in A$, we have $\int_B |f_\alpha| \, d\mu < \varepsilon$.*

Proof: Suppose $\{f_\alpha\}_{\alpha \in A}$ is uniformly integrable; with ε, λ_ε as in Definition 5.1, we have

$$I(|f_\alpha|) = \int_{\{|f_\alpha|>\lambda_\varepsilon\}} |f_\alpha| \, d\mu + \int_{\{|f_\alpha|\leq\lambda_\varepsilon\}} |f_\alpha| \, d\mu \leq \varepsilon + \lambda_\varepsilon \cdot \mu(\Omega) < \infty, \quad \forall \, \alpha \in A,$$

and for any $B \in \mathcal{F}$ with $\mu(B) < \delta_\varepsilon := \varepsilon/\lambda_\varepsilon$:

$$\begin{aligned} \int_B |f_\alpha| \, d\mu &= \int_{B\cap\{|f_\alpha|>\lambda_\varepsilon\}} |f_\alpha| \, d\mu + \int_{B\cap\{|f_\alpha|\leq\lambda_\varepsilon\}} |f_\alpha| \, d\mu \\ &\leq \int_{\{|f_\alpha|>\lambda_\varepsilon\}} |f_\alpha| \, d\mu + \lambda_\varepsilon \cdot \mu(B) < \varepsilon + \delta_\varepsilon \, \lambda_\varepsilon < 2\varepsilon . \end{aligned}$$

Now suppose that (a), (b) hold; from (a) and the Čebyšev inequality of (2.14), we obtain $\mu(|f_\alpha| > \lambda) \leq I(|f_\alpha|) / \lambda \leq K/\lambda$ for all $\alpha \in A$, $\lambda > 0$. Thus, for any $\delta > 0$ and $\lambda \geq (K+1)/\delta$, and all $\alpha \in A$, we have $\sup_{\alpha \in A} \mu(|f_\alpha| > \lambda) < \delta$. Substituting in (b) we obtain $\sup_{\alpha \in A} \int_{\{|f_\alpha|>\lambda\}} |f_\alpha| \, d\mu < \varepsilon$, for all $\lambda \geq \lambda_\varepsilon := (K+1)/\delta_\varepsilon$.

5.17 Exercise: Each of the following conditions is sufficient for the uniform integrability of the family $\{f_\alpha\}_{\alpha\in A}$ of measurable functions:

(i) there exists $f \in \mathbf{L}^1$ such that, for every $\alpha \in A$, we have: $|f_\alpha| \le |f|$, μ–a.e.

(ii) *Criterion of De la Vallée Poussin:* there exists a function $h : [0,\infty) \to [0,\infty)$ with $h(x)/x$ positive and increasing on $(0,\infty)$ and $\lim_{x\to\infty}(h(x)/x) = \infty$, such that: $\sup_{\alpha\in A}\int_\Omega (h \circ |f_\alpha|)\, d\mu < \infty$.

(iii) $\sup_{\alpha\in A}\int_\Omega |f_\alpha|^p\, d\mu < \infty$, for some $p > 1$.

5.18 Exercise: Show by example that a family of measurable functions can be

(i) bounded in $\mathbf{L}^1$ but not uniformly integrable;

(ii) uniformly integrable, but not dominated by an integrable function, as was the case in Exercise 5.17 (i).

The following result complements nicely the Lebesgue Dominated Convergence Theorem for functions in $\mathbf{L}^1$, by substituting the domination condition with uniform integrability.

5.2 Theorem: Generalized Dominated Convergence. *For a sequence of functions $\{f_n\}_{n\in\mathbf{N}} \subset \mathbf{L}^1$ which converges in measure to some measurable function f, the following conditions are equivalent:*

(i) $\{f_n\}_{n\in\mathbf{N}}$ *is uniformly integrable,*

(ii) $I(|f_n - f|) \longrightarrow 0$, as $n \to \infty$,

(iii) $I(|f_n|) \longrightarrow I(|f|) < \infty$, as $n \to \infty$.

Proof: $(i) \Rightarrow (ii)$. From Exercise 5.5 (v),(vi), Fatou's lemma and condition (a) of Proposition 5.1, we have $I(|f|) \le \liminf_k I(|f_{n_k}|) \le \sup_n I(|f_n|) =: K < \infty$. Then using Proposition 5.1 once again we see that the sequence $\{f_n - f\}_{n\in\mathbf{N}}$ is uniformly integrable, and

$$I(|f_n - f|) = \int_{\{|f_n - f| \le \varepsilon\}} |f_n - f|\, d\mu + \int_{\{|f_n - f| > \varepsilon\}} |f_n - f|\, d\mu$$

$$\le \varepsilon\, \mu(\Omega) + \int_{\{|f_n - f| > \varepsilon\}} |f_n - f|\, d\mu .$$

By assumption $\mu(|f_n - f| > \varepsilon) \to 0$, so the last integral also tends to zero as $n \to \infty$ (recall Proposition 5.1 (b)). Therefore $\limsup_n I(|f_n - f|) \le \varepsilon\, \mu(\Omega)$, and the result follows by letting $\varepsilon \downarrow 0$.

The implication $(ii) \Rightarrow (iii)$ follows directly from $|I(|f_n|) - I(|f|)| \le I(|f_n - f|) \to 0$, as $n \to \infty$ (triangle inequality). For the final implication $(iii) \Rightarrow (i)$, fix $\lambda > 0$ and introduce a bounded, uniformly continuous function $\varphi_\lambda : \mathbf{R} \to [0,\infty)$ with $\varphi_\lambda(x) \le |x|$, $\forall\, x \in \mathbf{R}$ and $\varphi_\lambda(x) = |x|$ for $|x| \le \lambda$, $\varphi_\lambda(x) = 0$ for $|x| > \lambda + 1$. Thus

$$\int_{\{|f| \le \lambda\}} |f|\, d\mu \le I(\varphi_\lambda(|f|)) = \lim_n I(\varphi_\lambda(|f_n|)) \le \liminf_n \int_{\{|f_n| \le \lambda + 1\}} |f_n|\, d\mu ,$$

thanks to Exercise 5.5.(vi). Subtracting memberwise from (iii), we obtain then

$$\limsup_n \int_{\{|f_n|>\lambda+1\}} |f_n| \, d\mu \le \int_{\{|f|>\lambda\}} |f| \, d\mu \longrightarrow 0 \,, \qquad \text{as } \lambda \to \infty \,.$$

The uniform integrability of $\{f_n\}_{n\in\mathbf{N}}$ follows now easily. $\diamondsuit$