1.6. PRODUCT MEASURE-SPACES

Let us consider a family $\{(\Omega_\alpha, \mathcal{F}_\alpha)\}_{\alpha \in A}$ of *nonempty* measurable spaces, as well as their Cartesian *product space* $\Omega := \prod_{\alpha \in A} \Omega_\alpha$; this is the set of all mappings $\omega : A \to \cup_{\alpha \in A} \Omega_\alpha$ such that $\omega_\alpha \equiv \omega(\alpha) \in \Omega_\alpha$ for every $\alpha \in A$. The so-called *Axiom of Choice* postulates that Ω is nonempty, when $A \neq \emptyset$. We would like to endow this product space Ω with a σ-algebra of measurable sets.

For this purpose, we introduce the class $\mathcal{C} := \{\pi_\alpha^{-1}(E_\alpha) \,|\, E_\alpha \in \mathcal{F}_\alpha,\, \alpha \in A\}$ of measurable *cylinder sets*, and define the *product σ-algebra* $\mathcal{F} \equiv \bigotimes_{\alpha \in A} \mathcal{F}_\alpha := \sigma(\mathcal{C})$ (collection of product-measurable sets) as the σ-algebra generated by the class of measurable cylinder sets. Here $\pi_\alpha : \Omega \to \Omega_\alpha$ is the α^{th} coördinate (projection) map $\pi_\alpha(\omega) = \omega(\alpha)$.

6.1 Remark : In the case of countable index-set A, the product σ-algebra $\mathcal{F}$ is also generated by the class of measurable *rectangles* $\mathcal{R} := \{\prod_{\alpha \in A} E_\alpha \,|\, E_\alpha \in \mathcal{F}_\alpha,\, \alpha \in A\}$, to wit, $\mathcal{F} = \sigma(\mathcal{R})$.

6.1 Exercise : If each $\mathcal{F}_\alpha = \sigma(\mathcal{E}_\alpha)$ is generated by a class $\mathcal{E}_\alpha$ of subsets of Ω_α, then the product $\sigma-$algebra can be expressed as $\mathcal{F} = \sigma(\mathcal{C}')$, where $\mathcal{C}' := \{\pi_\alpha^{-1}(E_\alpha) \,|\, E_\alpha \in \mathcal{E}_\alpha,\, \alpha \in A\}$. Furthermore, if A is countable and $\Omega_\alpha \in \mathcal{E}_\alpha$, $\forall\, \alpha \in A$, then we also have $\mathcal{F} = \sigma(\mathcal{R}')$, where $\mathcal{R}' := \{\prod_{\alpha \in A} E_\alpha \,|\, E_\alpha \in \mathcal{E}_\alpha,\, \alpha \in A\}$.

6.2 Exercise : Suppose that $\Omega_1, \cdots, \Omega_n$ are metric spaces, and equip their (finite) product space $\Omega = \prod_{j=1}^n \Omega_j$ with the product metric. Show then that $\bigotimes_{j=1}^n \mathcal{B}(\Omega_j) \subseteq \mathcal{B}(\Omega)$, with equality if the spaces $\Omega_1, \cdots, \Omega_n$ are separable. In particular, $\mathcal{B}(\mathbf{R}^n) = \mathcal{B}(\mathbf{R}) \otimes \cdots \otimes \mathcal{B}(\mathbf{R})$ is the $n-$fold product of the Borel σ-algebra on the real line.

Consider now the simplest case of just *two* component measure spaces $(\Omega_j, \mathcal{F}_j, \mu_j)$, $j = 1, 2$; on their product space $(\Omega, \mathcal{F}) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ we should like to define a **product measure** $\mu \equiv \mu_1 \otimes \mu_2$ with the property

$$\mu(E_1 \times E_2) = \mu_1(E_1) \cdot \mu_2(E_2), \qquad \forall\;\; E_1 \in \mathcal{F}_1,\; E_2 \in \mathcal{F}_2. \tag{6.1}$$

One way to go about this, is to follow the same path that we used for the construction of Lebesgue-Stieltjes measures in subsection 1.4.A: to wit, start by constructing the "proto-measure" $\ell(E) = \mu_1(E_1) \cdot \mu_2(E_2)$ on the elementary family $\mathcal{R}$ of measurable rectangles $E = E_1 \times E_2$ ($E_1 \in \mathcal{F}_1$, $E_2 \in \mathcal{F}_2$); obtain a pre-measure ν by extending ℓ in the obvious (and consistent) way to the algebra $\mathcal{E}$ of finite unions of disjoint such rectangles; and then extend this pre-measure to a measure μ on the product σ-algebra $\mathcal{F} = \sigma(\mathcal{E})$, while still satisfying the property (6.1).

Another approach, in fact the one that we shall follow, uses integration theory to obtain an **integral representation** of the product measure (cf. Theorem 6.1 below), and then exploits this representation to study the properties of integration with respect to this measure (the Tonelli-Fubini Theorems 6.2, 6.3). In order to describe these results, let us introduce the "section notation"

$$E_{\omega_i} := \{\omega_j \in \Omega_j \,|\, (\omega_1, \omega_2) \in E\} \quad \text{and} \quad f_{\omega_i}(\omega_j) := f(\omega_1, \omega_2)\,, \quad \omega_j \in \Omega_j \quad (j \neq i) \tag{6.2}$$

for subsets E of the product space Ω and for functions $f : \Omega \to \mathbf{R}$, with $\omega_i \in \Omega_i$ fixed.

It can be checked that *if* $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$ *, and if the mapping* f *is* $(\mathcal{F}_1 \otimes \mathcal{F}_2)-$*measurable, then the section* E_{ω_i} *is in* $\mathcal{F}_j$ *and the function* $\omega_j \mapsto f_{\omega_i}(\omega_j)$ *is* $\mathcal{F}_j-$*measurable, for* $j \neq i$. Indeed, the collection $\mathcal{G}$ of subsets E of $\Omega_1 \times \Omega_2$ with

$$E_{\omega_1} \in \mathcal{F}_2 \quad \text{for all} \;\; \omega_1 \in \Omega_1\,, \qquad \text{and} \qquad E_{\omega_2} \in \mathcal{F}_1 \quad \text{for all} \;\; \omega_2 \in \Omega_2$$

contains all measurable rectangles $E = E_1 \times E_2$ $(E_1 \in \mathcal{F}_1\,,\; E_2 \in \mathcal{F}_2\,)$, and is a σ-algebra: $\left(\bigcup_{n \in \mathbf{N}} E^{(n)}\right)_{\omega_i} = \bigcup_{n \in \mathbf{N}} (E^{(n)})_{\omega_i}\,,\; (E^c)_{\omega_i} = (E_{\omega_i})^c$ for $i = 1, 2$; consequently, $\mathcal{G} \supseteq \mathcal{F} \otimes \mathcal{F}_2$. The second claim then follows, since $(f_{\omega_i})^{-1}(B) = (f^{-1}(B))_{\omega_i}$ for $i = 1, 2$. Furthermore, the mapping

$$\omega_i \mapsto g_i(\omega_i) \equiv \mu_j(E_{\omega_i}) \quad \text{is} \;\; \mathcal{F}_i - \text{measurable, for} \;\; j \neq i\,, \tag{6.3}$$

at least when both spaces are σ-finite, as we shall see below (cf. proof of Theorem 6.1).

To illustrate the integral representation methods for constructing the product measure, let us recall from elementary calculus the computation of the area of the unit disc by means of a *single* integral.

6.1 Example: Let $\Omega_i = \mathbf{R}$, $\mathcal{F}_i = \mathcal{B}(\mathbf{R})$ with $\mu_i = \lambda$ Lebesgue measure $(i = 1, 2)$ and consider the unit circle $E = \{(\omega_1, \omega_2) \,|\, \omega_1^2 + \omega_2^2 \leq 1\} \in \mathcal{B}(\Omega)$ in the product-space $\Omega = \mathbf{R}^2$. Then in the notation of (6.2) and (6.3) we have: $E_{\omega_1} = \{\omega_2 : |\omega_2| \leq \sqrt{1 - \omega_1^2}\,\}$, $g_1(\omega_1) = \mu_2(E_{\omega_1}) = 2\sqrt{1 - \omega_1^2}$ for $|\omega_1| \leq 1$; and $E_{\omega_1} = \emptyset$, $g_1(\omega_1) = \mu_2(E_{\omega_1}) = 0$ for $|\omega_1| > 1$. The resulting function $g_1(\cdot)$ is clearly measurable, and its integral is $\int_{\Omega_1} g_1 \, d\mu_1 = 2 \int_{-1}^{1} \sqrt{1 - \omega_1^2} \, d\omega_1 = \pi$, the area of the unit disc.

The calculation of this example can be generalized, in a way that leads directly to a measure on the product σ-algebra with the desired property (6.1).

6.1 THEOREM : PRODUCT MEASURE. *If the component measure spaces are σ-finite, then the set-function*

$$\mu(E) := \int_{\Omega_1} g_1 \, d\mu_1 = \int_{\Omega_1} \mu_2(E_{\omega_1}) \, d\mu_1(\omega_1) \, , \quad E \in \mathcal{F}$$

is a σ-finite measure on the product σ-algebra $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$; it satisfies

$$\mu(E) = \int_{\Omega_2} g_2 \, d\mu_2 = \int_{\Omega_2} \mu_1(E_{\omega_2}) \, d\mu_2(\omega_2)$$

for every $E \in \mathcal{F}$, thus also the property (6.1); and is the unique *measure on $\mathcal{F}$ with the property (6.1).*

The measure of Theorem 6.1 is denoted $\mu = \mu_1 \otimes \mu_2$ and is called **product measure** of μ_1, μ_2 on $\mathcal{F}_1 \otimes \mathcal{F}_2$. Clearly, $\mu(E) = 0$, if and only if: $\mu_j(E_{\omega_i}) = 0$ for μ_i-a.e. $\omega_i \in \Omega_i$ $(i \neq j)$. And μ is a probability measure, if both μ_1, μ_2 are probability measures. The following two fundamental results describe the properties of integration with respect to this product-measure, first for positive and then for general, real-valued functions on Ω.

6.2 THEOREM : TONELLI. *In the context of Theorem 6.1, let $f : \Omega \to [0, \infty)$ be $\mathcal{F}-$measurable. Then the functions $\omega_i \mapsto h_i(\omega_i) := \int_{\Omega_j} f_{\omega_i} \, d\mu_j$ are $\mathcal{F}_i-$measurable, for $1 \leq i \neq j \leq 2$, and we have*

$$\int_{\Omega} f \, d\mu = \int_{\Omega_1} h_1 \, d\mu_1 = \int_{\Omega_2} h_2 \, d\mu_2 \tag{6.5}$$

or, more suggestively,

$$\begin{aligned} \int\int_{\Omega_1 \times \Omega_2} f(\omega_1, \omega_2) \; d(\mu_1 \otimes \mu_2)(\omega_1, \omega_2) \;&=\; \int_{\Omega_1} \left(\int_{\Omega_2} f_{\omega_1}(\omega_2) \, d\mu_2(\omega_2) \right) \, d\mu_1(\omega_1) \\ &=\; \int_{\Omega_2} \left(\int_{\Omega_1} f_{\omega_2}(\omega_1) \, d\mu_1(\omega_1) \right) \, d\mu_2(\omega_2) \, . \end{aligned}$$

6.3 THEOREM : FUBINI. *In the context of Theorem 6.1, consider a function $f \in \mathbf{L}^1(\Omega, \mathcal{F}, \mu)$, that is, integrable on the product-space.*

Then the function $f_{\omega_i}(\cdot)$ belongs to $\mathbf{L}^1(\Omega_j, \mathcal{F}_j, \mu_j)$ for μ_i-a.e. $\omega_i \in \Omega_i$, and the function h_i belongs to $\mathbf{L}^1(\Omega_i, \mathcal{F}_i, \mu_i)$, for $1 \leq i \neq j \leq 2$. Furthermore, the identities of (6.5) hold.

PROOF OF THEOREM 6.1 : Consider first the finite case $\mu_1(\Omega_1) + \mu_2(\Omega_2) < \infty$. We shall verify that the family

$$\mathcal{M} := \{ E \subseteq \Omega_1 \times \Omega_2 \;\; | \;\; \omega_1 \mapsto \mu_2(E_{\omega_1}) = g_1(\omega_1) \;\; \text{is} \;\; \mathcal{F}_1 - \text{measurable} \, \}$$

contains all product-measurable sets. Indeed, $\mathcal{M}$ contains the elementary class $\mathcal{R}$ of measurable rectangles, as well as the algebra $\mathcal{E}$ of finite disjoint unions of such rectangles, in the notation of the paragraph following (6.1). On the other hand, the continuity properties (2.5) and (2.15) of the measure μ_2 from below and above, respectively (the latter needs the assumption $\mu_2(\Omega_2) < \infty$), allow one to check that $\mathcal{M}$ is also a *monotone class* (Exercise 3.7). Thus, from the Monotone Class Theorem, $\mathcal{M} \supseteq m(\mathcal{E}) = \sigma(\mathcal{E}) =: \mathcal{F}_1 \otimes \mathcal{F}_2$.

• We can verify now that *the set-function*

$$\mu(E) := \int_{\Omega_1} g_1 \, d\mu_1 \equiv \int_{\Omega_1} \mu_2(E_{\omega_1}) \, d\mu_1(\omega_1) \,, \qquad E \in \mathcal{F}_1 \otimes \mathcal{F}_2$$

is a finite measure which satisfies (6.1) on $\mathcal{R}$.

To see this latter property, just note that for $E = E_1 \times E_2$ with $E_i \in \mathcal{F}_i$ $(i = 1, 2)$ we have: $E_{\omega_1} = E_2$ (resp., $\emptyset$) for $\omega_1 \in E_1$ (resp., for $\omega_1 \notin E_1$), and thus

$$\mu_2(E_{\omega_1}) = \chi_{E_1}(\omega_1) \cdot \mu_2(E_2) \,, \quad \mu(E) = \int_{\Omega_1} \mu_2(E_{\omega_1}) \, d\mu_1(\omega_1) = \mu_1(E_1) \cdot \mu_2(E_2) \,.$$

The uniqueness claim follows from the uniqueness part of the Hahn extension theorem. To check countable additivity, take any sequence $\{E^{(n)}\}_{n \in \mathbf{N}} \subseteq \mathcal{F}_1 \otimes \mathcal{F}_2$ of disjoint sets, set $E = \bigcup_{n \in \mathbf{N}} E^{(n)}$, observe that $E_{\omega_1} = \bigcup_{n \in \mathbf{N}} (E^{(n)})_{\omega_1}$ is again a disjoint union, and

$$\mu(E) \;=\; \int_{\Omega_1} \mu_2(E_{\omega_1}) \, d\mu_1(\omega_1) \;=\; \sum_{n \in \mathbf{N}} \int_{\Omega_1} \mu_2\big((E^{(n)})_{\omega_1}\big) \, d\mu_1(\omega_1) \;=\; \sum_{n \in \mathbf{N}} \mu(E^{(n)}) \,.$$

Interchanging the rôles of the two indices in the preceding argument, we see that the set-function $\widetilde{\mu}(E) := \int_{\Omega_2} \mu_1(E_{\omega_2}) \, d\mu_2(\omega)$ is a finite measure on $\mathcal{F}_1 \otimes \mathcal{F}_2$ that shares all the properties of μ; from uniqueness, $\widetilde{\mu} \equiv \mu$.

• If μ_1, μ_2 are only $\sigma-$finite, we can write $\Omega_1 \times \Omega_2$ as a countable, increasing union of disjoint rectangles $\Omega_1^{(n)} \times \Omega_2^{(n)}$ in $\mathcal{R}$, whose sides have finite measures. It suffices then to establish the result on *each* such rectangle, which we have already done; then pass to the limit, invoking the Monotone Convergence Theorem. $\diamond$

PROOF OF THEOREM 6.2 : If $f = \chi_E$ for some $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, then $h_i = g_i$ and the result reduces to Theorem 6.1. Thus the result holds for simple functions. For general $f \in \mathbf{L}^+$, let $\{f^{(n)}\}_{n \in \mathbf{N}} \subset \mathbf{L}^+$ increase pointwise to f; then $h_i^{(n)} \uparrow h_i$ pointwise (in particular, h_i is measurable), and $\int_{\Omega_i} h_i \, d\mu_i = \lim_n \int_{\Omega_i} h_i^{(n)} \, d\mu_i = \lim_n \int_\Omega f^{(n)} \, d\mu = \int_\Omega f \, d\mu$, $i = 1, 2$. $\diamond$

PROOF OF THEOREM 6.3 : If the function $f \in \mathbf{L}^1(\mu_1 \otimes \mu_2)$ is non-negative, we have $h_i < \infty$, μ_i-a.e. (that is, $f_{\omega_i} \in \mathbf{L}^1(\mu_j)$ for μ_i-a.e. $\omega_i \in \Omega_i$), as well as $h_i \in \mathbf{L}^1(\mu_i)$, for each $i \neq j$. In the general case, the result follows by applying Tonelli's theorem to each of f^+ , f^- separately. $\diamond$

6.2 Remark: It is straightforward to extend the above results to several dimensions: if $(\Omega_i, \mathcal{F}_i, \mu_i)$, $i = 1, \cdots, d$ are $\sigma-$finite measure spaces, then there is a unique measure μ on the product σ-algebra $\mathcal{F} = \bigotimes_{i=1}^d \mathcal{F}_i$, such that

$$\mu\,(E_1 \times \cdots \times E_d) \;=\; \mu_1(E_1) \cdots \mu_d(E_d)\,, \quad E_i \in \mathcal{F}_i\,,\; i = 1, \cdots, d\,. \tag{6.1^d}$$

This measure is denoted $\mu = \otimes_{i=1}^d \mu_i$ and is called the **product measure of** $\mu_1, \cdots, \mu_d$. Similarly, $\left(\prod_{i=1}^d \Omega_i, \bigotimes_{i=1}^d \mathcal{F}_i, \otimes_{i=1}^d \mu_i \right)$ is then called the **product measure space** of $(\Omega_i, \mathcal{F}_i, \mu_i)$, $i = 1, \cdots, d$. With the obvious modifications in notation, this measure has the properties set out in Theorems 6.1-6.3. In addition to the commutativity property

$$(\mu_2 \otimes \mu_1)(\mathcal{D}E) \;=\; (\mu_1 \otimes \mu_2)(E) \qquad \text{with } \;\mathcal{D}E := \{\, (\omega_1, \omega_2) \,|\, (\omega_2, \omega_1) \in E\,\}\,, \quad \text{for } E \in \mathcal{E}$$

implied by Theorem 6.1, the product measure is also associative:

$$\mu_1 \otimes (\mu_2 \otimes \mu_3) \;=\; (\mu_1 \otimes \mu_2) \otimes \mu_3\,.$$

6.3 Remark: If we take $(\Omega_i, \mathcal{F}_i) \equiv (\mathbf{R}, \mathcal{B}(\mathbf{R}))$, $\forall\; i = 1, \cdots, d$ above, and $F_i(\cdot) = \mu_i((-\infty, \cdot])$ is the distribution function on $\mathbf{R}$ corresponding to the measure μ_i , then the distribution function $F(\cdot) = \mu((-\infty, \cdot])$ induced on $\mathbf{R}^d$ by the product-measure μ (notation of §1.4.C), coincides with the product distribution function of Definition 4.3.

♣ The Tonelli-Fubini Theorems 6.2, 6.3 are most usefully invoked "in concatenation", in order to justify **inverting the order of integration** in double integrals of the form

$$\int_{\Omega_1} \int_{\Omega_2} f \, d\mu_1 \, d\mu_2 \;=\; \int_{\Omega} f \, d(\mu_1 \otimes \mu_2)\,.$$

Typically, one verifies *first* that $\int_\Omega |f| \, d(\mu_1 \otimes \mu_2)$ is finite, using Tonelli's theorem to evaluate this as a double integral – and *then* one applies Fubini's theorem to conclude $\int_{\Omega_1} \left(\int_{\Omega_2} f \, d\mu_2 \right) d\mu_2 = \int_{\Omega_2} \left(\int_{\Omega_1} f \, d\mu_1 \right) d\mu_2$.

As a (very good) rule of thumb, whenever you come across a double integral, *invert the order of integration!* Just do it; then worry about justifying what you did, using Theorems 6.2 and 6.3 as explained above. The following Propositions and Exercises illustrate the situation; for additional illustrations, see the proof of Theorem 2.2.2.

6.1 Proposition: Boundedness of Linear Operators on $\mathbf{L}^p$−spaces. *Let* $(X, \mathcal{F}, \mu)$ *and* $(Y, \mathcal{G}, \nu)$ *be σ-finite measure spaces, and* $K : X \times Y \to \mathbf{R}$ *an* $(\mathcal{F} \otimes \mathcal{G})$−*measurable function. Suppose that, for some* $C \in [0, \infty)$ *, we have*

(i) $\int_X |K(x,y)|\, d\mu(x) \le C$, *for* ν−*a.e.* $y \in Y$,

(ii) $\int_Y |K(x,y)|\, d\nu(y) \le C$, *for* μ−*a.e.* $x \in X$.

Then for every $f \in \mathbf{L}^p(\nu)$, $1 \le p \le \infty$, *the integral*

$$(Tf)(x) := \int_Y K(x,y) f(y)\, d\nu(y)$$

converges absolutely for μ−*a.e.* $x \in X$ *; the function* Tf *is well-defined and in* $\mathbf{L}^p(\mu)$*; and we have the* **Generalized Young's Inequality**

$$||Tf||_p \le C\, ||f||_p \; .$$

Proof: With $1 < p < \infty$ and $(1/p) + (1/q) = 1$, Hölder's inequality gives that $|Tf(x)|$ is dominated by

$$\begin{aligned} \int_Y |K(x,y) f(y)|\, d\nu(y) &= \int_Y |K(x,y)|^{1/q} \cdot |K(x,y)|^{1/p}\, |f(y)|\, d\nu(y) \\ &\le \left(\int_Y |K(x,y)|\, d\nu(y) \right)^{1/q} \cdot \left(\int_Y |K(x,y)|\, |f(y)|^p\, d\nu(y) \right)^{1/p} \\ &\le C^{1/q} \left(\int_Y |K(x,y)|\, |f(y)|^p\, d\nu(y) \right)^{1/p} \end{aligned}$$

for μ−a.e. $x \in X$ and, by Tonelli's theorem, $\int_X |Tf(x)|^p\, d\mu(x)$ is dominated by

$$\int_X \left(\int_Y |K(x,y) f(y)|\, d\nu(y) \right)^p d\mu(x) \le C^{(p/q)} \int_X \int_Y |K(x,y)|\; |f(y)|^p d\nu(y)\, d\mu(x)$$

$$\le C^{(p/q)} \int_Y |f(y)|^p \left(\int_X |K(x,y)|\, d\mu(x) \right) d\nu(y) \le C^{1+(p/q)} \int_Y |f(y)|^p\, d\nu(y) < \infty \, .$$

Now Fubini's theorem implies that, for μ−a.e. $x \in X$, the mapping $y \mapsto K(x,y) f(y)$ is in $\mathbf{L}^1(\nu)$ and so $Tf(x)$ is well-defined; furthermore, $\int_X |Tf(x)|^p\, d\mu(x) \le C^p\, (||f||_p)^p$.

For $p = 1$ a similar proof works, which uses only (i); for $p = \infty$ the conclusion is immediate, and relies only on (ii). ◇

6.3 EXERCISE : CONVOLUTION, FOURIER TRANSFORM, AND THE YOUNG INEQUALITY. For any two measurable functions $f : \mathbf{R}^d \to \mathbf{R}$ and $g : \mathbf{R}^d \to \mathbf{R}$, the **convolution** $f * g$ of f and g is the function defined by

$$(f * g)(x) := \int_{\mathbf{R}^d} f(x-y)\, g(y)\, dy = \int_{\mathbf{R}^d} g(x-y)\, f(y)\, dy \tag{6.6}$$

for all $x \in \mathbf{R}^d$ such that the integral on the right-hand-side is well-defined and finite. For instance, if $f \in \mathbf{L}^p(\mathbf{R}^d)$ and $q \in \mathbf{L}^q(\mathbf{R}^d)$ with $p \geq 1$, $(1/p) + (1/q) = 1$, then the Hölder inequality guarantees that $(f * g)(x)$ is well-defined and finite for every $x \in \mathbf{R}^d$.
(i) Assuming that all integrals in question exist, show that

$$f * g = g * f \,, \quad \big(f * g\big) * h \;=\; f * \big(g * h\big) \,, \quad \text{and} \quad \mathrm{supp}\big(f * g\big) \subseteq \mathrm{supp}\big(f\big) + \mathrm{supp}\big(g\big) \,,$$

where $K + L := \{x + y \,;\, x \in K \,,\ y \in L\}$. We are denoting here by $\mathrm{supp}(f)$ the *support* of the function f, that is, the smallest closed set outside of which the function vanishes.
(ii) Show that, for every $g \in \mathbf{L}^1(\mathbf{R}^d)$ and $f \in \mathbf{L}^p(\mathbf{R}^d)$ for some $1 \leq p \leq \infty$, the convolution $(f * g)(x)$ of (6.6) is well-defined for $\lambda-$a.e. $x \in \mathbf{R}^d$, and satisfies **Young's inequality**

$$\|f * g\|_p \;\leq \|g\|_1 \, \|f\|_p \;.$$

(iii) With $i = \sqrt{-1}$, the **Fourier Transform** of $f \in \mathbf{L}^1(\mathbf{R}^d)$ is the function $\widehat{f} : \mathbf{R}^d \to \mathbf{C}$ defined by

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{\,i\langle \xi, x\rangle}\, f(x)\, dx \,, \qquad \xi \in \mathbf{R}^d \,. \tag{6.7}$$

Show that $\widehat{f}(\cdot)$ is uniformly continuous, and uniformly bounded: $\sup_{\xi \in \mathbf{R}^d} |\widehat{f}(\xi)| \leq \|f\|_1 < \infty$. Show also that *the Fourier transform of the convolution is the product of the Fourier transforms,* in the sense that

$$\widehat{f * g} \;=\; \widehat{f} \cdot \widehat{g} \quad \text{holds for every} \quad f \in \mathbf{L}^1(\mathbf{R}^d) \,,\ g \in \mathbf{L}^1(\mathbf{R}^d) \,.$$

6.2 Proposition: Minkowski Inequality for Integrals. *Let* $(X, \mathcal{F}, \mu)$ *and* $(Y, \mathcal{G}, \nu)$ *be* σ*-finite measure spaces, and* $f : X \times Y \to \mathbf{R}$ *an* $(\mathcal{F} \otimes \mathcal{G})-$*measurable function.*
(i) If $f \geq 0$ *and* $1 \leq p < \infty$*, we have*

$$\left[\int_X \left(\int_Y f(x,y)\, d\nu(y)\right)^p d\mu(x)\right]^{1/p} \leq \int_Y \left[\int_X \big(f(x,y)\big)^p\, d\mu(x)\right]^{1/p} d\nu(y) \;=:\; F \,.$$

(ii) Suppose that $1 \leq p \leq \infty$*, and that the function* $y \mapsto \|f(\cdot, y)\|_p$ *is in* $\mathbf{L}^1(\nu)$*. Then* $f(x, \cdot) \in \mathbf{L}^1(\nu)$ *for* $\mu-$*a.e.* $x \in X$*, the function* $x \mapsto \int_Y f(x,y)\, d\nu(y)$ *is in* $\mathbf{L}^p(\mu)$*, and we have the* Minkowski-type inequality

$$\left\| \int_Y f(\cdot, y)\, d\nu(y) \right\|_p \leq \int_Y \|f(\cdot, y)\|_p \, d\nu(y) \;=\; F \,.$$

Proof: If $p=1$, then part (i) is just Tonelli's theorem. For $1<p<\infty$, part (i) is again obvious, if $F=\infty$; with $F<\infty$, let $(1/p)+(1/q)=1$, take $g \in \mathbf{L}^q(\mu)$ and observe that

$$\int_X \left[\int_Y f(x,y)\, d\nu(y) \right] |g(x)|\, d\mu(x) = \int_Y \left[\int_X f(x,y)\, |g(x)|\, d\mu(x) \right] d\nu(y)$$
$$\leq \int_Y \left[||g||_q \cdot \left(\int_X (f(x,y))^p \; d\mu(x) \right)^{1/p} \right] d\nu(y)$$

from Tonelli's theorem and the Hölder inequality. In other words, with the notation $h(x) := \int_Y f(x,y)\, d\nu(y)$ we have

$$I^\mu(h|g|) = \int_X h(x)\, |g(x)|\, d\mu(x) \leq F \cdot ||g||_q\,, \quad \forall\, g \in \mathbf{L}^q(\mu)\,.$$

Now the "converse Hölder inequality" of Exercise 5.10 (ii) gives $h \in \mathbf{L}^p(\mu)$ and $||h||_p \leq F$, which is part (i) of the Proposition; whereas Fubini's theorem (with f replaced by $|f|$) yields part (ii). For $p=\infty$, part (ii) follows from the monotonicity of the integral. ⋄

6.4 EXERCISE : LAYERED REPRESENTATION. Let ν be a measure on $\mathcal{B}(\mathbf{R})$ with $N(u) := \nu([0,u)) < \infty$, $\forall\, u>0$, and suppose that $g : \Omega \to [0,\infty)$ is a Borel-measurable function on the $\sigma-$finite measure space $(\Omega, \mathcal{F}, \mu)$. Then with the shorthand notation $\mu(g>u) \equiv \mu\left(\{\omega \in \Omega \,|\, g(\omega)>u\}\right)$ already used in (2.14), show that

$$\int_\Omega N(g(\omega))\, d\mu(\omega) = \int_{[0,\infty)} \mu(g>u)\, d\nu(u)\,. \tag{6.8}$$

In particular, if $d\nu(u) = pu^{p-1}du$ for some $p>0$, then

$$\int_\Omega (g(\omega))^p\, d\mu(\omega) = p \int_0^\infty u^{p-1}\, \mu(g>u)\, du\,. \tag{6.9}$$

6.5 Exercise : Potential Theory on the Line. For any measure μ on $\mathcal{B}(\mathbf{R})$, define its *potential*

$$(\mathcal{P}\mu)(x) := \int_{\mathbf{R}} |x-y|\, d\mu(y)\,, \quad x \in \mathbf{R}\,. \tag{6.10}$$

(a) Show that this recipe defines a convex function $\mathcal{P}\mu$, which is real-valued provided that we have $\int_{\mathbf{R}} \big(1+|y|\big)\, d\mu(y) < \infty$.

(b) Suppose that μ is a probability measure with $\int_{\mathbf{R}} |y|\, d\mu(y) < \infty$, $\int_{\mathbf{R}} y\, d\mu(y) = 0$, and denote by $F(\cdot) := \mu\left((-\infty, \cdot]\right)$ its distribution function and by $\delta_0(A) := \chi_A(0)$, $A \in \mathcal{B}(\mathbf{R})$ the *Dirac measure* at the origin. Show that $\mathcal{P}\mu - \mathcal{P}\delta_0$ is nonnegative and tends to zero as

$|x| \to \infty$; furthermore, the area between the two curves $(\mathcal{P}\mu)(\cdot)$ and $(\mathcal{P}\delta_0)(\cdot)$ is equal to the 'second moment' $\int_{-\infty}^{\infty} y^2\, d\mu(y)$ of the measure μ. More precisely, we have

$$(\mathcal{P}\mu)(x) \;=\; \int_{-\infty}^{x} F(y)\, dy \;+\; \int_{x}^{\infty} (1 - F(y))\, dy\,, \qquad x \in \mathbf{R}\,,$$

$$\begin{aligned}(\mathcal{P}\mu)(x) - (\mathcal{P}\delta_0)(x) \;=\; (\mathcal{P}\mu)(x) - |x| \;&=\; 2\int_{x}^{\infty} (1 - F(y))\, dy\,, \quad \text{for} \;\; x \geq 0\,,\\ &=\; 2\int_{-\infty}^{x} F(y)\, dy\,, \qquad \text{for} \;\; x \leq 0\,.\end{aligned}$$

and

$$\int_{-\infty}^{\infty} (\mathcal{P}\mu - \mathcal{P}\delta_0)(x)\, dx \;=\; \int_{-\infty}^{\infty} y^2\, d\mu(y)\,.$$

6.6 Exercise : Burkholder's Moderate-Function Inequality. A function $F : (0,\infty) \to (0,\infty)$ is called *moderate,* if it is continuous, increasing with $F(0+) = 0$, $F(\infty) = \infty$, and satisfies

$$\sup_{x>0} \left(\frac{F(\alpha x)}{F(x)} \right) < \infty\,, \qquad \text{for some (and then for every)} \;\; \alpha > 1\,. \tag{6.11}$$

Suppose that f, g are non-negative measurable functions on a finite measure space $(\Omega, \mathcal{F}, \mu)$, that satisfy

$$\mu\,(\, f > \beta\lambda\,,\; g \leq \delta\lambda\,) \;\leq\; \psi(\delta) \cdot \mu(f > \lambda)\,; \quad \forall\, \delta > 0\,,\; \lambda > 0 \tag{6.12}$$

for some $\beta > 1$ and some $\psi : (0,\infty) \to (0,\infty)$ continuous, increasing with $\psi(0+) = 0$. Show that for every moderate function F, there exists a real constant $C \equiv C_{\beta,\psi,F}$ (which does *not* depend on f or g), such that

$$I(F(f)) \;\leq\; C \cdot I(F(g))\,. \tag{6.13}$$

(*Hint:* Integrate both sides of (6.12) with respect to $dF(\lambda)$, and use Fubini-Tonelli.)

6.7 Exercise : Completion of the Product Space. Suppose that the two component measure spaces $(\Omega_i, \mathcal{F}_i, \mu_i)$, $i = 1, 2$ in Theorems 6.1-6.3 are complete. Then the product space $(\Omega, \mathcal{F}, \mu) \equiv (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$ is *not necessarily complete.* Nonetheless, one can replace in these results the product measure space by its completion $(\Omega, \bar{\mathcal{F}}, \bar{\mu})$ as in Exercise 3.4, and the conclusions will still hold.

A: SMOOTH FUNCTIONS, APPROXIMATION OF THE IDENTITY

6.1 Definition : Smooth Functions. For every $n \in \mathbf{N}$ we shall denote by $C^n(\mathbf{R})$ the space of functions $f : \mathbf{R} \to \mathbf{R}$ which are continuous along with their derivatives $D^m f(\cdot) \equiv f^{(m)}(\cdot)$ of all orders $m = 1, \cdots, n$. The intersection $C^\infty(\mathbf{R}) := \bigcap_{n\in\mathbf{N}} C^n(\mathbf{R})$ is then the space of **infinitely differentiable functions.**

Similarly, for every $n \in \mathbf{N}$ we shall denote by $C^n_\downarrow(\mathbf{R})$ the subset of $C^n(\mathbf{R})$ that consists of "rapidly decreasing functions", i.e., satisfying $\lim_{|x|\to\infty} \left(x^k D^m f(x)\right) = 0$ for all $k \in \mathbf{N}$, $m = 0, 1, \cdots, n$.

The set $C^\infty_\downarrow(\mathbf{R}) := \bigcap_{n\in\mathbf{N}} C^n_\downarrow(\mathbf{R})$ will be called the **Schwartz space** of infinitely differentiable, rapidly decreasing functions. Clearly, for every $f \in C^\infty_\downarrow(\mathbf{R})$, we have $D^m f \in \mathbf{L}^p(\mathbf{R})$ for all $1 \le p < \infty, m \in \mathbf{N}_0$.

Finally, we shall denote by $C^n_*(\mathbf{R})$, $C^\infty_*(\mathbf{R})$ the subsets of $C^n_\downarrow(\mathbf{R})$, $C^\infty_\downarrow(\mathbf{R})$, respectively, that consist of functions with 'compact support' (that is, with $f \equiv 0$ outside a compact set $K \subset \mathbf{R}$).

Observe that the function $x \mapsto e^{-x^2}$ belongs to the Schwartz space $C^\infty_\downarrow(\mathbf{R})$, but the function $x \mapsto 1/(1+x^2)$ does not; neither does the function $x \mapsto e^{-|x|}$, though for a different reason. The next exercise establishes the 'smoothing properties' of the convolution operation in (6.6).

6.8 Exercise : Approximation to the Identity. Given a function $\varphi \in \mathbf{L}^1(\mathbf{R}^d)$ with $||\varphi||_1 = 1$, consider $\varphi_\varepsilon(x) \equiv \varepsilon^{-1}\varphi\left(x\,\varepsilon^{-1}\right)$, $x \in \mathbf{R}$ for each $\varepsilon > 0$, so that $||\varphi_\epsilon||_1 = 1$.

(i) For every $f \in \mathbf{L}^p(\mathbf{R}^d)$ with $1 \le p < \infty$, we have in the notation of (6.6) for the convolution: $(f * \varphi_\varepsilon) \longrightarrow f$ in $\mathbf{L}^p(\mathbf{R}^d)$ as $\varepsilon \downarrow 0$.

(ii) If $f \in \mathbf{L}^\infty(\mathbf{R}^d)$ is uniformly continuous on a set B, then $(f * \varphi_\varepsilon) \longrightarrow f$ uniformly on B, as $\varepsilon \downarrow 0$.

(iii) Suppose that $\varphi \in C^\infty_\downarrow(\mathbf{R}^d)$, and consider $f \in \mathbf{L}^p(\mathbf{R}^d)$ for some $1 \le p \le \infty$. Then $(f * \varphi) \in C^\infty(\mathbf{R}^d)$, and

$$D^m(f * \varphi) = f * \left(D^m \varphi\right), \quad \forall\, m \in \mathbf{N}_0. \tag{6.14}$$

These same conditions also hold for f locally in $\mathbf{L}^p(\mathbf{R}^d)$, under the stronger assumption $\varphi \in C^\infty_*(\mathbf{R}^d)$.

6.9 Exercise : Approximation by Smooth Functions. The space $C^\infty_*(\mathbf{R})$ of infinitely differentiable functions with compact support is dense in $\mathbf{L}^p(\mathbf{R})$, $\forall\ 1 \le p < \infty$.

6.10 Exercise : Separability of $\mathbf{L}^p(\mathbf{R}^d)$. There exists a sequence $\{h_n\}_{n\in\mathbf{N}}$ of simple functions on $\mathbf{R}^d$, with the following property: For each $1 \le p < \infty$, each $\Omega \in \mathcal{B}(\mathbf{R}^d)$, each $f \in \mathbf{L}^p(\mathbf{R}^d)$, and each $\varepsilon > 0$, we have: $||f - h_k||_p < \varepsilon$ for some $k \in \mathbf{N}$.

B: PROBABILITY MEASURES ON INFINITE-DIMENSIONAL SPACES

Let us tackle now the question of constructing measures on infinite-dimensional spaces. We start with an infinite set $\mathbf{T}$ (countable or not); for each $t \in \mathbf{T}$ we let $\Omega_t = \mathbf{R}$, $\mathcal{F}_t = \mathcal{B}(\mathbf{R})$, and consider the *canonical* space $\Omega := \prod_{t \in \mathbf{T}} \Omega_t \equiv \mathbf{R}^{\mathbf{T}}$, consisting of all real-valued functions $\omega : \mathbf{T} \to \mathbf{R}$ on $\mathbf{T}$. We also consider the class $\mathcal{C}^*$ of *finite-dimensional cylinder sets,* i.e., sets of the form

$$C = \{\omega \in \Omega \,|\, (\omega(t_1), \cdots, \omega(t_n)) \in A\} \quad \text{with } A \in \mathcal{B}(\mathbf{R}^n)\,, \quad n \in \mathbf{N} \tag{6.15}$$

as well as the $\sigma-$algebra $\mathcal{F} := \sigma(\mathcal{C}^*)$.

Now let $\mathcal{T}_n$ denote the set of finite sequences $\tau = (t_1, \cdots, t_n)$, $n \in \mathbf{N}$ of *distinct* $n-$tuples of elements in $\mathbf{T}$, and set $\mathcal{T} := \cup_{n \in \mathbf{N}} \mathcal{T}_n$. Suppose that, for each $n \in \mathbf{N}$ and $\tau \in \mathcal{T}_n$, we have prescribed a probability distribution function $F_\tau : \mathbf{R}^n \to [0, 1]$ with corresponding Lebesgue-Stieltjes measure $\mu_\tau \equiv \mu_{F_\tau}$ on $\mathcal{B}(\mathbf{R}^n)$. We say that $\{F_\tau\}_{\tau \in \mathcal{T}}$ (respectively, $\{\mu_\tau\}_{\tau \in \mathcal{T}}$) is a family of *finite-dimensional probability distribution functions* (resp., of *finite-dimensional distributions*).

Here is the question of interest: *given a family $\{F_\tau\}_{\tau \in \mathcal{T}}$ as above, can we construct a probability measure $\mathbf{P}$ on $(\Omega, \mathcal{F})$ so that*

$$\mathbf{P}[\,\omega \in \Omega \,|\, (\omega(t_1), \cdots, \omega(t_n)) \in A\,] \,=\, \mu_\tau(A)\,, \quad \forall\, A \in \mathcal{B}(\mathbf{R}^n) \tag{6.16}$$

holds for every $\tau = (t_1, \cdots, t_n) \in \mathcal{T}$ and $n \in \mathbf{N}$? In other words, can we put together a probability measure $\mathbf{P}$ on $\Omega = \mathbf{R}^{\mathbf{T}}$ when we are given all its finite-dimensional "marginal" distributions $\{F_\tau\}_{\tau \in \mathcal{T}}$?

If such a measure $\mathbf{P}$ exists, then it is fairly straightforward to see from (6.16) that the following two **Consistency Conditions** *(C.C.'s)* have to be satisfied, for every $n \in \mathbf{N}$:

(C.C.1) If $\varsigma = (t_{i_1}, \cdots, t_{i_n})$ is a permutation of $\tau = (t_1, \cdots, t_n) \in \mathcal{T}_n$, then for any Borel subsets $A_1, \cdots, A_n$ of the real line we have: $\mu_\tau \left(\prod_{j=1}^n A_j \right) = \mu_\varsigma \left(\prod_{j=1}^n A_{i_j} \right)$.

(C.C.2) If $\tau = (t_1, \cdots, t_n) \in \mathcal{T}_n$ and $\varsigma = (t_1, \cdots, t_n, t_{n+1})$, then $\mu_\tau(B) = \mu_\varsigma(B \times \mathbf{R})$, for any $B \in \mathcal{B}(\mathbf{R}^n)$.

The following result asserts that these conditions are not only necessary, but also *sufficient* for the existence of such a probability measure $\mathbf{P}$. The proof uses in a crucial manner the regularity of each of the finite-dimensional distributions μ_τ, $\tau = (t_1, \cdots, t_n) \in \mathcal{T}_n$ on $\mathcal{B}(\mathbf{R}^n)$ for $n \in \mathbf{N}$, as in Exercise 1.5.

6.4 THEOREM : DANIELL-KOLMOGOROV. *Let $\{F_\tau\}_{\tau\in\mathcal{T}}$ be a given family of finite-dimensional p.d.f.'s, and suppose that the family of corresponding finite-dimensional distributions $\{\mu_\tau\}_{\tau\in\mathcal{T}}$ satisfies the Consistency Conditions (C.C.1), (C.C.2) above. Then there exists a probability measure $\mathbf{P}$ on the canonical space $(\Omega,\mathcal{F})$, such that (6.16) holds.*

6.2 Example: Let $\{F_n\}_{n\in\mathbf{N}}$ be a sequence of probability distribution functions on the real line (with corresponding Lebesgue-Stieltjes measures $\mu_n \equiv \mu_{F_n}$, $n\in\mathbf{N}$). We let $\mathbf{T}=\mathbf{N}$, and consider the product probability distribution function

$$F_\tau(x_1,\cdots,x_n) := F_{t_1}(x_1)\cdots F_{t_n}(x_n)\,,\quad \forall\ (x_1,\cdots,x_n)\in\mathbf{R}^n$$

for any $\tau=(t_1,\cdots,t_n)\in\mathcal{T}$ (recall Definition 4.3), with associated Lebesgue-Stieltjes measure $\mu_\tau \equiv \mu_{F_\tau} = \bigotimes_{j=1}^n \mu_{t_j}$ on $\mathcal{B}(\mathbf{R}^n)$. It is clear that the family $\{\mu_\tau\}_{\tau\in\mathbf{T}}$ satisfies the Consistency Conditions of Theorem 6.4. According to this result, there exists a probability measure $\mathbf{P}$ on the canonical space $(\Omega,\mathcal{F})\equiv(\mathbf{R}^{\mathbf{N}},\sigma(\mathcal{C}^*))$ such that

$$\begin{aligned}\mathbf{P}[\,\omega\in\Omega\,|\,\omega(t_1)\in A_1\,,\cdots,\omega(t_n)\in A_n\,] \;&=\; \mu_{t_1}(A_1)\cdots\mu_{t_n}(A_n) \\ &=\prod_{j=1}^n \mathbf{P}[\,\omega\in\Omega\,|\,\omega(t_j)\in A_j\,]\end{aligned}\tag{6.17}$$

holds for any Borel subsets $A_1,\cdots,A_n$ of the real line, $n\in\mathbf{N}$, and $(t_1,\cdots,t_n)\in\mathcal{T}$.

Under this probability measure, the coördinate mappings $X_n(\omega):=\omega_n$, $n\in\mathbf{N}$ are independent random variables with prescribed (one-dimensional marginal) distributions $\mathbf{P}[\,X_n\le x\,]=F_n(x)$, $x\in\mathbf{R}$.

PROOF OF THEOREM 6.4: For any cylinder set $C\in\mathcal{C}^*$ of the form (6.15), we set $\mathbf{P}(C)=\mu_\tau(A)$ where $\tau=(t_1,\cdots,t_n)\in\mathcal{T}_n$ and $A\in\mathcal{B}(\mathbf{R}^n)$. We leave it as an excercise, to check that the two consistency conditions (C.C.1), (C.C.2) guarantee $\mathbf{P}$ is well-defined and finitely-additive on $\mathcal{C}^*$ by this recipe, and $\mathbf{P}(\Omega)=1$. If we can show that $\mathbf{P}$ is also *countably additive* on $\mathcal{C}^*$, then the Carathéodory-Hahn Theorems 3.1, 3.3 will guarantee that $\mathbf{P}$ can be extended to a probability measure on $\mathcal{F}=\sigma(\mathcal{C}^*)$.

To this end, suppose that $\{B_k\}_{k\in\mathbf{N}}$ are disjoint sets in $\mathcal{C}^*$ with $B:=\cup_{k\in\mathbf{N}}B_k\in\mathcal{C}^*$, set $C_m := B\setminus(\cup_{k=1}^m B_k)$ so that $\mathbf{P}(B)=\mathbf{P}(C_m)+\sum_{k=1}^m\mathbf{P}(B_k)$ for each $m\in\mathbf{N}$, and observe $\cap_{m\in\mathbf{N}}C_m=\emptyset$. Countable additivity will follow, as soon as we manage to show

$$\ell := \lim_{m\to\infty}\mathbf{P}(C_m) \;=\; 0\,.\tag{6.18}$$

The sequence $\{C_m\}_{m\in\mathbf{N}}$ is decreasing, so the limit in (6.18) exists. We shall assume that $\ell>0$, and try to arrive at a contadiction.

Step 1: Monotonicity. There exists a decreasing sequence $\{D_m\}_{m\in\mathbf{N}} \subset \mathcal{C}^*$ with the property $\cap_{m\in\mathbf{N}} D_m = \emptyset$ and $\lim_{m\to\infty} \mathbf{P}(D_m) = \ell > 0$, of the form

$$D_m = \{\omega \in \Omega \,|\, (\omega(t_1), \cdots, \omega(t_m)) \in A_m\} \qquad \text{for some } \; A_m \in \mathcal{B}(\mathbf{R}^m)\,,$$

such that $\tau_m = (t_1, \cdots, t_m) \in \mathcal{T}_m$ is an extension of $(t_1, \cdots, t_{m-1}) \in \mathcal{T}_{m-1}$ for every $m \geq 2$.

To see this, observe that since $C_{k+1} \subseteq C_k$, each of the sets C_k is of the form: $C_k = \{\omega \in \Omega \,|\, (\omega(t_1), \cdots, \omega(t_{m_k})) \in A_{m_k}\}$ for some $A_{m_k} \in \mathcal{B}(\mathbf{R}^{m_k})$, such that $A_{m_{k+1}} \subseteq A_{m_k} \times \mathbf{R}^{m_{k+1}-m_k}$ and $(t_1, \cdots, t_{m_{k+1}})$ is an extension of $(t_1, \cdots, t_{m_k})$. Define

$$D_1 = \{\omega \in \Omega \,|\, \omega(t_1) \in \mathbf{R}\}\;, \cdots, D_{m_1-1} = \left\{\omega \in \Omega \,|\, (\omega(t_1), \cdots, \omega(t_{m_1-1})) \in \mathbf{R}^{m_1-1}\right\}$$

and $D_{m_1} = C_1$; then $D_{m_1+1} = \{\omega \in \Omega \,|\, (\omega(t_1), \cdots, \omega(t_{m_1}), \omega(t_{m_1+1})) \in A_{m_1} \times \mathbf{R}\}$, $\cdots$, $D_{m_2-1} = \left\{\omega \in \Omega \,|\, (\omega(t_1), \cdots, \omega(t_{m_1}), \omega(t_{m_1+1}), \cdots, \omega(t_{m_2-1})) \in A_{m_1} \times \mathbf{R}^{m_2-m_1-1}\right\}$ and $D_{m_2} = C_2$. Continuing this procedure, we see $\cap_{m\in\mathbf{N}} D_m = \cap_{m\in\mathbf{N}} C_m = \emptyset$.

Step 2: Regularity. From Exercise 1.5, there exists a closed set $F_m \subseteq A_m$ such that $\mu_{\tau_m}(A_m \setminus F_m) < \varepsilon\, 2^{-m}$ for every $m \in \mathbf{N}$. Intersect this F_m with a sufficiently large closed ball to obtain a compact set K_m such that

$$E_m := \{\omega \in \Omega \,|\, (\omega(t_1), \cdots, \omega(t_m)) \in K_m\} \subseteq D_m\,, \qquad \mathbf{P}(D_m \setminus E_m) = \mu_{\tau_m}(A_m \setminus K_m) < \frac{\varepsilon}{2^m}\,.$$

This sequence $\{E_m\}_{m\in\mathbf{N}}$ may not be decreasing, so we define $\widetilde{E}_m = \cap_{k=1}^m E_k$ and note that $\widetilde{E}_m = \{\omega \in \Omega \,|\, (\omega(t_1), \cdots, \omega(t_m)) \in \widetilde{K}_m\}$ with

$$\widetilde{K}_m \;=\; \left(K_1 \times \mathbf{R}^{m-1}\right) \cap \left(K_2 \times \mathbf{R}^{m-2}\right) \cap \cdots (K_{m-1} \times \mathbf{R}) \cap K_m$$

a compact set and $\mu_{\tau_m}(\widetilde{K}_m) = \mathbf{P}(\widetilde{E}_m) > 0$, because

$$\begin{aligned}\mathbf{P}(\widetilde{E}_m) \;&=\; \mathbf{P}(D_m) - \mathbf{P}(D_m \setminus \widetilde{E}_m) \;=\; \mathbf{P}(D_m) - \mathbf{P}\left(\cup_{k=1}^m (D_m \setminus E_k)\right) \\ &\geq\; \mathbf{P}(D_m) - \mathbf{P}\left(\cup_{k=1}^m (D_k \setminus E_k)\right) \;\geq\; \ell - \sum_{k=1}^m \frac{\ell}{2^k} \;>\; 0\,.\end{aligned}$$

Step 3: Diagonalization. We have just shown that $\widetilde{K}_m$ is non-empty, so we may choose an element $\left(x_1^{(m)}, \cdots, x_m^{(m)}\right) \in \widetilde{K}_m$ for every $m \in \mathbf{N}$. The resulting sequence $\left\{x_1^{(m)}\right\}_{m\in\mathbf{N}}$ is contained in the compact set $\widetilde{K}_1$, so it must contain a subsequence $\left\{x_1^{(m_k)}\right\}_{k\in\mathbf{N}}$ that converges to some $x_1 \in \widetilde{K}_1$. By the same token, $\left\{\left(x_1^{(m_k)}, x_2^{(m_k)}\right)\right\}_{k\in\mathbf{N}}$ is a sequence in the compact set $\widetilde{K}_2$, so it too contains a subsequence that converges to some $(x_1, x_2) \in \widetilde{K}_2$. Continuing this way we can put together a sequence of real numbers $(x_1, x_2, \cdots)$ such that $(x_1, x_2, \cdots, x_m) \in \widetilde{K}_m$ for each $m \in \mathbf{N}$. In other words,

$$S \;=\; \{\, \omega \in \Omega \,|\, \omega(t_i) = x_i\,,\; i \in \mathbf{N}\,\} \subset \widetilde{E}_m \subseteq D_m\,, \quad \forall\, m \in \mathbf{N}\,,$$

contradicting $\cap_{m\in\mathbf{N}} D_m = \emptyset$. This shows that (6.18) holds. $\diamond$