2.9. BROWNIAN MOTION

Consider a particle of pollen immersed in water and subjected to the buffeting of the surrounding water molecules. The particle undergoes an irregular, diffusive motion, first observed by the botanist R. Brown in 1828. This motion was then studied in 1905 by A. Einstein; he provided a physical theory that led to the first experimental computation of the Avogadro number and dispelled any remaining doubts about the molecular/atomic nature of matter.

A one-dimensional caricature of this motion can be constructed as follows: imagine that the particle undergoes a sequence of $I.I.D.$ Bernoulli "kicks" $\xi_1, \xi_2, \cdots$ with $\mathbf{P}[\xi_1 = \pm 1] = 1/2$ of size $h > 0$ at regular time-intervals of length $\delta > 0$. Thus, the location of the particle after n such kicks (that is, at time δn) is $h \sum_{j=1}^{n} \xi_j(\omega)$. More generally, the location of the particle at time t is given by the *Simple Random Walk*

$$S_0(\omega) = 0, \qquad S_t(\omega) \;=\; h \cdot \sum_{j=1}^{[t/\delta]} \xi_j(\omega) \qquad \text{for} \quad 0 < t < \infty$$

where $[x] := \max\{n \in \mathbf{Z} \,|\, n \le x\}$ is the 'integer-part' function. This is a piecewise constant, right-continuous random function; it has independent increments $S_{t_1}(\omega)\,,\, S_{t_2}(\omega) - S_{t_1}(\omega)\,, \cdots, S_{t_m}(\omega) - S_{t_{m-1}}(\omega)$ for $0 < t_1 < t_2 < \cdots < t_m < \infty$ that are multiples of δ. Clearly $\mathbf{E}(S_t) = 0$ and $\mathrm{Var}(S_t) = h^2\,[t/\delta]$.

We would like to get a continuous picture of the particle's movement by letting both the duration and the size of the kicks tend to zero, i.e., $h \downarrow 0$ and $\delta \downarrow 0$. But we must also be careful to obtain a limit that is both finite and non-degenerate. One way to accomplish this, is to ensure $h^2 = \sigma^2\,\delta$ for some real constant $\sigma > 0$; in particular, by taking $\delta_n = 1/n$, $h_n = \sigma/\sqrt{n}$ and looking at the *sequence of Simple Random Walks*

$$S_0^{(n)}(\omega) = 0, \qquad S_t^{(n)}(\omega) \;=\; \frac{\sigma}{\sqrt{n}} \cdot \sum_{j=1}^{[nt]} \xi_j(\omega) \qquad \text{for} \quad 0 < t < \infty$$

indexed by $n \in \mathbf{N}$. Recalling the Central Limit Theorems 5.1 and 5.3, we see the following properties:

(a) for fixed $0 < t < \infty$, the distributions of the random variables $\{S_t^{(n)}\,;\, n \in \mathbf{N}\}$ converge vaguely to the distribution $\mathcal{N}(0, \sigma^2 t)$ of a normal (Gaussian) random variable W_t with mean zero and variance $\sigma^2 t$;

(b) for fixed $m \in \mathbf{N}$ and $0 < t_1 < t_2 < \cdots < t_m < \infty$, the distributions of the random vectors

$$\Big(S_{t_1}^{(n)}(\omega)\,,\, S_{t_2}^{(n)}(\omega) - S_{t_1}^{(n)}(\omega)\,, \cdots, S_{t_m}^{(n)}(\omega) - S_{t_{m-1}}^{(n)}(\omega) \Big)\,, \qquad n \in \mathbf{N}$$

converge vaguely to the distribution of a vector $\big(W_{t_1}(\omega)\,,\, W_{t_2}(\omega) - W_{t_1}(\omega)\,, \cdots, W_{t_m}(\omega) - W_{t_{m-1}}(\omega)\big)$ of independent normal (Gaussian) random variables with mean zero and $\mathrm{Var}(W_{t_j}(\omega) - W_{t_{j-1}}) = \sigma^2\,(t_j - t_{j-1})$.

We can easily imagine now that the entire sequence $\big\{S_t^{(n)}\,;\, 0 \le t < \infty\big\}_{n\in\mathbf{N}}$ converges, in a suitable sense, to a family of random variables $\mathcal{W} = \{W_t\,,\, 0 \le t < \infty\}$ with the following properties:

(i) $W_0 \equiv 0$;

(ii) $W_t - W_s$ has normal (Gaussian) distribution $\mathcal{N}\big(0, \sigma^2(t-s)\big)$ with mean zero and variance $\sigma^2(t-s)$;

(iii) the increments $\big(W_{t_1}(\omega)\,,\, W_{t_2}(\omega) - W_{t_1}(\omega)\,, \cdots, W_{t_m}(\omega) - W_{t_{m-1}}(\omega)\big)$ are independent, for any $0 < t_1 < t_2 < \cdots < t_m < \infty$; and

(iv) $\mathbf{P}\,[\,\omega \in \Omega\,|\,$ the function $t \mapsto W_t(\omega)$ is continuous$\,] = 1$.

9.1 Definition: A family of random variables $\mathcal{W} = \{W_t\,,\, 0 \le t < \infty\}$ with the properties (i)-(iv) is called *Brownian Motion process*; it is also called *standard* when $\sigma = 1$.

This is a Gaussian family of random variables with mean and covariance functions

$$m(t) := \mathbf{E}(W_t) = 0 \qquad \text{and} \qquad \Sigma(t,s) := \mathbf{E}\,(W_t W_s) = t \wedge s\,, \tag{9.1}$$

respectively, since $\mathbf{E}\,(W_t W_s) = \mathbf{E}\left[W_s^2 + W_s\,(W_t - W_s)\right] = s$ for $0 < s < t < \infty$ by the independence of increments property; recall the Definition 2.3. And conversely, any collection of real-valued random variables $\{W_t\,,\ 0 \le t < \infty\}$ with a.e. path $t \mapsto W_t(\omega)$ continuous, which is a zero-mean Gaussian family with covariance as in (9.1), is Brownian motion.

The Definition 9.1 provides the probability distribution function of the random vector $\big(W_{s_1}, \cdots, W_{s_m}\big)$ for any $0 = s_0 < s_1 < \cdots < s_m < \infty$ as

$$F_{(s_1,\cdots,s_m)}(x_1, \cdots, x_m) \;:=\; \mathbf{P}\big[\,W_{s_1} \le x_1, \cdots W_{s_m} \le x_m\,\big] \tag{9.2}$$

$$= \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\cdots\int_{-\infty}^{x_m} p_{s_1}(0,y_1)\,p_{s_2-s_1}(y_1,y_2)\cdots p_{s_m-s_{m-1}}(y_{m-1},y_m)\,dy_m\cdots dy_2\,dy_1$$

for $(x_1, \cdots, x_m) \in \mathbf{R}^m$, $m \in \mathbf{N}$, where

$$p_t(x,y) \;:=\; \frac{1}{\sqrt{2\pi t}}\,e^{-(x-y)^2/2t}\,, \qquad t > 0\,,\ (x,y) \in \mathbf{R}^2 \tag{9.3}$$

is the *fundamental Gaussian kernel:* the transition probability density for the Brownian Motion process. It can be checked that the specification (9.2) is equivalent to the statement that the increments $\{W_{s_j} - W_{s_{j-1}}\}_{j=1}^m$ are independent and normally distributed with $\mathbf{E}\big(W_{s_j} - W_{s_{j-1}}\big) = 0$ and $\mathbf{E}\big(W_{s_j} - W_{s_{j-1}}\big)^2 = s_j - s_{j-1}$.

9.1 Exercise: Show that the recipe (9.2) induces a family $\{F_\tau\}_{\tau \in \mathcal{T}}$ of finite-dimensional distributions in the notation of §1.6.A, which satisfies the consistency conditions of Theorem 1.6.4.

9.2 Exercise: If $\mathcal{W} = \{W_t\,,\, 0 \le t < \infty\}$ is a family of random variables with independent increments (in particular, Brownian motion), show that $W_t - W_s$ is independent of the $\sigma-$algebra $\mathcal{F}_s = \sigma(W_u\,,\, 0 \le u \le s)$.

We shall deal with the question of constructing Brownian Motion in the next subsection. For the time being, we shall assume the existence of this process and try to establish some of its elementary properties. The first three of these are consequences of the Gaussian family characterization (9.1) of Brownian motion, whereas the fourth follows directly from the second.

9.1 Proposition: Invariance under Scaling and Symmetry. *The process* $c^{-1/2}\, W_{ct}\,,\ 0 \le t < \infty$ *is Brownian Motion, for any given* $c > 0$. *Similarly, the process* $-W_t\,,\ 0 \le t < \infty$ *is Brownian Motion.*

9.2 Proposition: Invariance under Time-Inversion. *The process* $\widetilde{\mathcal{W}}$ *defined by* $\widetilde{W}_0 = 0\,,\ \widetilde{W}_t = t\, W_{1/t}$ *for* $0 < t < \infty$, *is Brownian Motion.*

Proof: Only the continuity of $\widetilde{\mathcal{W}}$ is at issue here, because the processes $\{\, W_t\,,\, 0 < t < \infty \,\}$ and $\{\, \widetilde{W}_t\,,\, 0 < t < \infty \,\}$ are both continuous (on $(0, \infty)$), and both of them are zero-mean Gaussian families with the covariance structure of (9.1):

$$\mathbf{E}\big(\widetilde{W}_t \widetilde{W}_s\big) \;=\; ts \cdot \min\left(\frac{1}{t}\,,\, \frac{1}{s}\right) \;=\; \min(t,s)\,.$$

But for this reason the event

$$\widetilde{F} \;:=\; \left\{ \lim_{t \downarrow 0} \widetilde{W}_t = 0 \right\} \;=\; \bigcap_{n \in \mathbf{N}} \bigcup_{m \in \mathbf{N}} \bigcap_{\substack{q \in \mathbf{Q} \\ 0 < q \le 1/m}} \left\{ \big|\widetilde{W}_q\big| \le \frac{1}{n} \right\}$$

has exactly the same probability as the event

$$\bigcap_{n \in \mathbf{N}} \bigcup_{m \in \mathbf{N}} \bigcap_{\substack{q \in \mathbf{Q} \\ 0 < q \le 1/m}} \left\{ \big|W_q\big| \le \frac{1}{n} \right\} \;=\; \left\{ \lim_{t \downarrow 0} W_t = 0 \right\} \;=:\; F\,,$$

namely $\mathbf{P}\big(\widetilde{F}\big) \;=\; \mathbf{P}(F) \;=\; 1$, where the last equality is from the fact that $\mathcal{W}$ is Brownian Motion. $\diamond$

9.3 Proposition: Invariance under Time-Reversal. *For fixed* $T \in (0, \infty)$ *, the process* $\widetilde{W}_t = W_T - W_{T-t}$ *,* $0 < t \leq T$ *is Brownian Motion on* $[0, T]$ *.*

9.4 Proposition: Strong Law of Large Numbers. *We have* $\lim_{t\to\infty} (W_t(\omega)/t) = 0$ *for a.e.* $\omega \in \Omega$ *.*

Indeed, from Proposition 9.2 (read with $\tau = 1/t$) we obtain: $W_\tau(\omega)/\tau = \widetilde{W}_{1/\tau}(\omega) \to \widetilde{W}_0(\omega) = 0$ as $\tau \to \infty$, for a.e. $\omega \in \Omega$.

9.5 Proposition: Markov Property. *For every* $t \in (0, \infty)$ *the process* $\mathcal{B}$ *defined by* $B_u = W_{t+u} - W_t$ *,* $0 \leq u < \infty$ *is Brownian Motion, and is independent of the* $\sigma-$*algebra* $\mathcal{F}_t := \sigma(W_s,\, 0 \leq s \leq t)$. In particular, for every bounded, Borel-measurable $f : \mathbf{R} \to \mathbf{R}$ we have

$$\mathbf{E}\big(f(W_{t+u}) \,|\, \mathcal{F}_t\big) \;=\; (\Pi_u f)(W_t)\,, \qquad \text{a.e.} \tag{9.4}$$

where $\{\Pi_u\}_{u\geq 0}$ is the *Brownian transition semigroup* defined as $\Pi_0 := f$ and

$$(\Pi_u f)(x) \;:=\; \int_{\mathbf{R}} p_u(x,y)\, f(y)\, dy\,, \qquad 0 < u < \infty\,. \tag{9.5}$$

The semigroup property

$$\Pi_{t+u} \;=\; \Pi_t \cdot \Pi_u \;=\; \Pi_u \cdot \Pi_t\,, \qquad u \geq 0\,,\; t \geq 0 \tag{9.6}$$

follows directly from (9.3), and is also known as *Chapman-Kolmogorov equation.*

9.3 Exercise: Infinitesimal Generator. Show that the *infinitesimal generator*

$$\mathcal{G}f \;:=\; \lim_{t\downarrow 0} \frac{1}{t}\,(\Pi_t f - f) \tag{9.7}$$

of the Brownian transition semigroup $\{\Pi_t\}_{t\geq 0}$ in (9.5) is given as

$$\mathcal{G}f \;=\; \frac{1}{2}\,\frac{\partial^2 f}{\partial x^2}\,, \qquad \text{at least for } \; f \in \mathcal{C}_b^2(\mathbf{R})\,.$$

Deduce that for any such f (twice continuously differentiable, with f, f' and f'' all bounded), the function $u(t,x) := (\Pi_t f)(x) = \mathbf{E}[f(x + W_t)]$ satisfies the *heat equation*

$$\frac{\partial u}{\partial t} \;=\; \frac{1}{2}\,\frac{\partial^2 u}{\partial x^2}\,; \qquad t > 0\,,\; x \in \mathbf{R} \tag{9.8}$$

and the initial condition $u(0,x) = f(x)\,,\; x \in \mathbf{R}$.

We can express the Markov property of Proposition 9.5 by saying that Brownian motion *starts afresh* at any fixed time $t \in (0, \infty)$. We shall see in Proposition 9.9 below that the Markov property also holds for a special and important class of random times, called **stopping times**.

9.6 Proposition: Unboundedness and Recurrence. *For a.e.* $\omega \in \Omega$ *we have*

$$\sup_{0 \le t < \infty} W_t(\omega) \;=\; \infty \,, \qquad\qquad \inf_{0 \le t < \infty} W_t(\omega) \;=\; -\infty$$

and thus, for every given $b \in \mathbf{R}$*, the set* $\{\, t \ge 0 \,|\, W_t(\omega) = b \,\}$ *is unbounded.*

In other words: Brownian Motion is **recurrent**; it visits every site on the real line, and keeps returning to it over and over.

Proof: The scaling property of Proposition 9.1 shows that $M := \sup_{0 \le t < \infty} W_t$ has the same distribution as cM for every $c \in (0, \infty)$, so the distribution of M is concentrated on the set $\{0, \infty\}$: $\mathbf{P}(M = 0) = \mathbf{P}(M = \infty) = 1$.

We have to show $\mathbf{P}(M = 0) = 0$. Indeed, $\mathbf{P}(M = 0)$ is dominated by

$$\mathbf{P}\Big(W_1 \le 0 \;\text{ and }\; W_s \le 0\,,\; \forall\, s \ge 1 \Big) \;\le\; \mathbf{P}\Big(W_1 \le 0 \;\text{ and }\; \sup_{t \ge 0} \big(W_{1+t} - W_1 \big) = 0 \Big)\,;$$

the inequality holds because $\widetilde{W}_t := W_{1+t} - W_1$, $0 \le t < \infty$ is Brownian Motion, whose supremum $\widetilde{M} = \sup_{0 \le t < \infty} \widetilde{W}_t$ is again either zero or infinity. This is a consequence of Proposition 9.5, which also asserts that $\widetilde{W}$ is independent of $\{W_u\,,\; 0 \le u \le 1\}$; we are then led to $p := \mathbf{P}(M = 0) \le \mathbf{P}(W_1 \le 0) \cdot \mathbf{P}\big(\widetilde{M} = 0\big) = \frac{1}{2} \cdot p$, thus $p = 0$. ◇

9.4 Exercise: Use Propositions 9.2, 9.6 to show that a.e. Brownian path $t \mapsto W_t(\omega)$ is non-differentiable at the origin $t = 0$; then argue that

$$\mathbf{P}\Big(\omega \in \Omega \,\Big|\, \text{the path } t \mapsto W_t(\omega) \text{ is non-differentiable at Lebesgue-a.e. } t \ge 0 \Big) \;=\; 1\,. \tag{9.9}$$

9.7 Proposition: Quadratic Variation. *Consider the dyadic rational partitions* $t_j^{(n)} = tj2^{-n}$ *of the interval* $[0, t]$*; for a.e.* $\omega \in \Omega$ *we have*

$$\lim_{n \to \infty} \sum_{j=1}^{2^n} \Big(W_{t_j^{(n)}}(\omega) - W_{t_{j-1}^{(n)}}(\omega) \Big)^2 \;=\; t \qquad \textit{and} \qquad \lim_{n \to \infty} \sum_{j=1}^{2^n} \Big| W_{t_j^{(n)}}(\omega) - W_{t_{j-1}^{(n)}}(\omega) \Big| \;=\; \infty\,.$$

Proof: Dropping the supersript on the time-indices, consider the random variables

$$\begin{aligned} D_n \;:=\; \sum_{j=1}^{2^n} \big(W_{t_j} - W_{t_{j-1}} \big)^2 - t \;&=\; \sum_{j=1}^{2^n} \Big\{ \big(W_{t_j} - W_{t_{j-1}} \big)^2 - (t_j - t_{j-1}) \Big\} \\ =\; \sum_{j=1}^{2^n} (t_j - t_{j-1}) \left[\left(\frac{W_{t_j} - W_{t_{j-1}}}{\sqrt{t_j - t_{j-1}}} \right)^2 - 1 \right] \;&=\; t\, 2^{-n} \cdot \sum_{j=1}^{2^n} \big(Z_j^2 - 1 \big)\;; \end{aligned}$$

here $Z_j = (W_{t_j} - W_{t_{j-1}})/\sqrt{t_j - t_{j-1}}$, $j = 1, \cdots, 2^n$ are independent copies of a standard normal random variable Z . Thus

$$\mathbf{E}\left(D_n^2\right) = \frac{t^2}{4^n} \cdot 2^n \, \mathbf{E}\left(Z^2 - 1\right)^2 = \text{const.} \frac{t^2}{2^n}, \qquad \text{and} \qquad \sum_{n \in \mathbf{N}} \mathbf{E}\left(D_n^2\right) < \infty .$$

It follows from Theorem 4.1 (iii) that $\lim_{n\to\infty} D_n(\omega) = 0$ for every $\omega \in \Omega^*$ with $\mathbf{P}(\Omega^*) = 1$, establishing the first claim. The second follows then from the continuity of $t \mapsto W_t(\omega)$ which implies $\max_{1 \le j \le 2^n} \left| W_{t_j}(\omega) - W_{t_{j-1}}(\omega) \right| \longrightarrow 0$ as $n \to \infty$, and the inequality

$$t + D_n(\omega) \le \max_{1 \le j \le 2^n} \left| W_{t_j}(\omega) - W_{t_{j-1}}(\omega) \right| \cdot \sum_{j=1}^{2^n} \left| W_{t_j^{(n)}}(\omega) - W_{t_{j-1}^{(n)}}(\omega) \right| . \qquad \diamond$$

In other words, almost every path $t \mapsto W_t(\omega)$ of the Brownian Motion process provides an example of a *continuous function with infinite variation on any bounded interval;* recall Exercise 1.4.10. One can also show that almost every function $t \mapsto W_t(\omega)$

- is nowhere differentiable on $(0, \infty)$;
- has no point of increase (or decrease) on $(0, \infty)$, namely

$$\mathbf{P}\Big(\text{there exist } t > 0 ,\ \delta \in (0, t) \text{ s.t. } W_{t-h} \le W_t \le W_{t+h} ,\ \forall\ h \in [0, \delta]\Big) = 1 ;$$

- is nowhere Hölder-continuous with exponent $\gamma > 1/2$; in particular,
- is nowhere Lipschitz continuous; but
- is locally Hölder continuous for any exponent $\gamma \in (0, 1/2)$, i.e.,

$$\sup_{\substack{0 < t-s < h(\omega) \\ s,t \in [0,1]}} \frac{|W_t(\omega) - W_s(\omega)|}{|t-s|^\gamma} \le L$$

for some random variable $h : \Omega \to (0, \infty)$ and an appropriate constant $L \in (0, \infty)$.

These properties suggest a picture of a continuous but highly irregular, jagged motion, which has indeed some resemblance to the diffusive particulate motion first observed by R. Brown – or to the movement of certain asset prices, in a different application first envisaged by L. Bachelier in 1900. The fine structure of the Brownian paths continues to be a fascinating subject, almost 200 years after R. Brown and more that 100 years after L. Bachelier.

More precisely, we have the following strengthening of (9.9).

9.8 PROPOSITION: Nowhere-differentiability of the Brownian Path (Dvoretzky, Erdös & Kakutani (1961)). *For a.e. $\omega \in \Omega$ the Brownian path $t \mapsto W_t(\omega)$ is nowhere Lipschitz-continuous. In particular,*

$$\mathbf{P}\Big(\omega \in \Omega \,\Big|\, \text{the path } t \mapsto W_t(\omega) \text{ is differentiable at some } t \ge 0\Big) = 0 . \qquad (9.9)'$$

Proof: For any given $L \in \mathbf{N}$, define the events

$$A_n^{(L)} := \left\{ |W_t - W_s| \le L\,|t-s| \ \text{ for some } t \in [0,1] \ \text{ and all } s \text{ such that } |t-s| \le 2/n \right\}$$

for $n \ge 2$, and note $A_n^{(L)} \subseteq \bigcup_{k=2}^{n} \left\{ |W_{j/n} - W_{(j-1)/n}| \le \frac{4L}{n} \quad \text{for } j = k-1, k, k+1 \right\}$. Therefore,

$$\frac{\mathbf{P}\big(A_n^{(L)}\big)}{n-2} \le \Big(\mathbf{P}\big(|W_{1/n} - W_0| \le 4L/n\big)\Big)^3 = \Big(\mathbf{P}\big(|W_1| \le 4L/\sqrt{n}\,\big)\Big)^3 \le \frac{\text{const}}{\sqrt{n^3}}\,;$$

since the sequence $\{A_n\}_{n \ge 2}$ is increasing, we get $\mathbf{P}\big(\bigcup_{n \ge 2} A_n^{(L)}\big) = \lim_{n \to \infty} \mathbf{P}\big(A_n^{(L)}\big) = 0$. The result follows now by observing

$$\{\omega \in \Omega \,|\, \text{ the path } t \mapsto W_t(\omega) \text{ is differentiable at some } t \ge 0\} \subseteq \bigcup_{L \in \mathbf{N}} \bigcup_{n \ge 2} A_n^{(L)}\,. \qquad \diamond$$

9.5 Exercise: Let $\mathcal{W}$ be standard Brownian Motion and X an independent normal random variable with $\mathbf{E}(X) = m$ and $\mathrm{Var}(X^2) = \sigma^2 > 0$. Show that the processes below are all Brownian Motions:

$$\widetilde{W}_t = W_t + Xt - \int_0^t \frac{W_s + Xs + \frac{m}{\sigma^2}}{s + \frac{1}{\sigma^2}}\, ds\,, \qquad \widehat{W}_t = W_t - \int_0^t \frac{W_s}{s}\, ds\,; \qquad 0 \le t < \infty$$

$$\overline{W}_t = W_t - \int_0^t \frac{W_1 - W_s}{1-s}\, ds\,, \qquad 0 \le t \le 1.$$

9.6 Exercise: Show that a.e. Brownian path $t \mapsto W_t(\omega)$ is monotone in no interval.

9.7 Exercise: Modify the proof of Proposition 9.8 to obtain the following stronger result: *For a.e.* $\omega \in \Omega$ *the Brownian path* $t \mapsto W_t(\omega)$ *is nowhere Hölder-continuous with exponent* $\gamma > 1/2$.

A: THE CONSTRUCTION OF BROWNIAN MOTION

From the Daniell-Kolmogorov Theorem 1.6.4 and Exercise 9.1, we know there exists a probability measure $\widetilde{\mathbf{P}}$ on the space $\widetilde{\Omega} = \mathbf{R}^{[0,\infty)}$ of *all* functions $\omega : [0,\infty) \to \mathbf{R}$ with $\omega(0) = 0$, under which the coördinate mapping process $W_t(\omega) = \omega(t)$, $0 \le t < \infty$ has independent, Gaussian increments, with mean zero and $\mathrm{Var}(W_t - W_s) = t - s$. But this space $\mathbf{R}^{[0,\infty)}$ *is not canonical for the Brownian Motion process*: it is not at all clear that the set $C([0,\infty)) \equiv \{\omega \in \widetilde{\Omega} \,|\, t \mapsto W_t(\omega) \text{ is continuous}\}$ belongs to the σ-algebra generated by the cylinder sets, let alone that it might be assigned full $\widetilde{\mathbf{P}}-$measure.

This is not a mere "technical snag"; it reflects the fact that it is not possible to determine a function in $\mathbf{R}^{[0,\infty)}$ simply by specifying its values at countably-many coördinates (whilst such a determination *is* possible for a function in $C([0,\infty))$). It also underscores the awkwardness inherent in the Daniell-Kolmogorov setup, which pairs the huge space $\mathbf{R}^{[0,\infty)}$ with the embarrassingly small $\sigma-$algebra $\sigma(\mathcal{C}^*)$ generated by its cylinder sets. The situation is really bad: the only $\sigma(\mathcal{C}^*)-$measurable set contained in $C([0,\infty))$ is the empty set!

Thus, the Brownian Motion Process of Definition 9.1 needs to be constructed from scratch. This was accomplished first by N. Wiener in 1928, via a *random Fourier series representation.* We shall present here a related but not identical approach based on Haar functions and Hilbert-space ideas, and due successively to Paley, Wiener, Lévy, Ciesielski, and most recently Pinsky (2001). This approach constructs Brownian Motion on the unit interval $[0,1]$; it is then straightforward to piece together independent copies of this construction and arrive at Brownian Motion on $[0,\infty)$. Wiener's original construction appears in Exercise 9.8.

The **Haar functions** provide a convenient total orthonormal basis for the Hilbert space $\mathcal{H} := \mathbf{L}^2([0,1])$ of square-integrable functions on the unit interval $[0,1]$. They are defined as $h_{00} \equiv 1$; $h_{01}(t) = 1$ for $0 \le t < 1/2$ and $h_{01}(t) = -1$ for $1/2 \le t \le 1$; and for $i \in \mathbf{N}$, $j = 1, 2, \cdots, 2^i$ by

$$h_{ij}(t) = 2^{i/2} \quad \text{for} \quad \frac{2j-2}{2^{i+1}} \le t < \frac{2j-1}{2^{i+1}}\,, \qquad h_{ij}(t) = -2^{i/2} \quad \text{for} \quad \frac{2j-1}{2^{i+1}} \le t < \frac{2j}{2^{i+1}}$$

and $h_{ij}(t) = 0$ otherwise. One checks the orthonormality property: $\int_0^1 h_{ij}^2(t)\,dt = 1$ and $\int_0^1 h_{ij}(t)\,h_{kl}(t)\,dt = 0$ for all $(i,j) \ne (k,l)$.

To show *totality*, take any $f \in \mathcal{H}$ with $\int_0^1 f(t)\,h_{ij}(t)\,dt = 0$ for all (i,j); in particular, $\int_0^1 f(t)\,dt = 0$ (for $i=0,\ j=0$) and $\int_0^{1/2} f(t)\,dt - \int_{1/2}^1 f(t)\,dt = 0$ (for $i=0,\ j=1$), so $\int_0^{1/2} f(t)\,dt = \int_{1/2}^1 f(t)\,dt = 0$. Now induction gives $\int_{(j-1)2^{-i}}^{j2^{-i}} f(t)\,dt = 0$ for all (i,j), and Lebesgue's Differentiation Theorem 1.4.1 shows that $f = 0$ holds $\lambda-$a.e. in $[0,1]$.

Then Theorem $B.3$ and Remark $B.2$ in Appendix B give us the Parseval Equation

$$||f||^2 = \int_0^1 f^2(t)\,dt = \sum_i \sum_j \left(\int_0^1 f(t)\,h_{ij}(t)\,dt \right)^2 = \sum_i \sum_j \left| \langle f, h_{ij} \rangle \right|^2 \qquad (9.10.a)$$

and its bilinear version for all f and g in $\mathcal{H}$:

$$\begin{aligned}\langle f,g\rangle \;=\; \int_0^1 f(t)\,g(t)\,dt \;&=\; \sum_i\sum_j \Big(\int_0^1 f(t)\,h_{ij}(t)\,dt\Big)\Big(\int_0^1 g(t)\,h_{ij}(t)\,dt\Big)\\ &=\; \sum_i\sum_j \langle f,h_{ij}\rangle\,\langle g,h_{ij}\rangle\,.\end{aligned} \tag{9.10.b}$$

Now we construct a complete probability space $(\Omega,\mathcal{F},\mathbf{P})$ with an array $\{Z_{ij}\}_{1\le i,j<\infty}$ of independent copies of the standard normal random variable Z with $\mathbf{P}[z\le x]=\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-u^2/2}\,du$. The random variable

$$W_t(\omega) \;=\; Z_{00}(\omega)\int_0^t h_{00}(s)\,ds \;+\; \sum_{i\in\mathbf{N}_0}\sum_{j=1}^{2^i} Z_{ij}(\omega)\int_0^t h_{ij}(s)\,ds \tag{9.11}$$

is well-defined because the series converges in $\mathbf{L}^2(\Omega,\mathcal{F},\mathbf{P})$. It is also checked that the mapping $[0,\infty)\times\Omega\ni(t,\omega)\mapsto W_t(\omega)\in\mathbf{R}$ is $\mathcal{B}\big([0,\infty)\big)\otimes\mathcal{F}-$measurable. *We claim that* $\mathcal{W}=\{W_t\,,\ 0\le t<\infty\}$ *is a standard Brownian Motion.*

Indeed, $\mathcal{W}$ is a Gaussian family as in Definition 2.3, with mean function $\mathbf{E}(W_t)=0$ and

$$\begin{aligned}\mathbf{E}\left(W_t^2\right) \;&=\; \Big(\int_0^t h_{00}(s)\,ds\Big)^2 \;+\; \sum_{i\in\mathbf{N}_0}\sum_{j=1}^{2^i}\Big(\int_0^t h_{ij}(s)\,ds\Big)^2\\ &=\; \big(\langle\chi_{[0,t]},h_{00}\rangle\big)^2 + \sum_{i\in\mathbf{N}_0}\sum_{j=1}^{2^i}\big(\langle\chi_{[0,t]},h_{ij}\rangle\big)^2 \;=\; \big|\big|\chi_{[0,t]}\big|\big|^2 \;=\; t\end{aligned}$$

from the Parseval identity (9.10.a). Slightly more generally, with $0\le u\le s<t\le 1$ we have $\mathbf{E}(W_t-W_s)=0$, as well as

$$\mathbf{E}(W_t-W_s)^2 = \big(\langle\chi_{[s,t]},h_{00}\rangle\big)^2 + \sum_{i\in\mathbf{N}_0}\sum_{j=1}^{2^i}\big(\langle\chi_{[s,t]},h_{ij}\rangle\big)^2 \;=\; \big|\big|\chi_{[s,t]}\big|\big|^2 \;=\; t-s\,,$$

$$\mathbf{E}\left[(W_t-W_s)\,W_u\right] \;=\; \sum_{i\in\mathbf{N}_0}\sum_{j=1}^{2^i}\langle\chi_{[s,t]},h_{ij}\rangle\,\langle\chi_{[0,u]},h_{ij}\rangle \;=\; \langle\chi_{[s,t]},\chi_{[0,u]}\rangle \;=\; 0$$

from the Parseval identity (9.10.b). But this leads to the correct covariance function specification $\Sigma(s,t)\;=\;\mathbf{E}[W_s\,W_t]\;=\;\mathbf{E}\left[(W_s)^2+W_s\,(W_t-W_s)\right]\;=\;s$ for the Brownian Motion. It remains to check the continuity requirement; this is done in the following result, which even hints at the modulus of continuity of the Brownian path.

9.1 Theorem: *For* $\mathbf{P}-$*a.e.* $\omega \in \Omega$ *, the function* $t \mapsto W_t(\omega)$ *of (9.11) is uniformly continuous: for a suitable random variable* $M : \Omega \to (0,\infty)$ *, we have*

$$\left|W_t(\omega) - W_s(\omega)\right| \le M(\omega) \cdot \sqrt{\delta \log\left(\frac{1}{\delta}\right)} \qquad \text{as long as} \qquad |t-s| \le \delta < 1\,. \tag{9.12}$$

We shall need a couple of elementary estimates before broaching the proof.

9.1 Lemma: *There exists a random variable* $M_1 : \Omega \to (0,\infty)$ *such that for a.e.* $\omega \in \Omega$ *:*

$$\sup_{i,j}\left(\frac{|Z_{ij}(\omega)|}{\sqrt{i}}\right) \le M_1(\omega)\,, \qquad\qquad \left|\sum_{j=1}^{2^i} Z_{ij}(\omega)\cdot\int_0^t h_{ij}(u)\,du\right| \le M_1(\omega)\cdot\sqrt{i}\,2^{-i/2}$$

Proof: Enumerate (i,j) by setting $N(0,0)=1$, $N(i,j) = 2^i + j \le 2^{i+1}$ for $j = 0,1,\cdots,2^i$ to obtain a singly-indexed sequence $\{Z_n\} \equiv \{Z_{N(i,j)}\}$ of independent, standard normal random variables. From (7.4) we have $\mathbf{P}(\,|Z_N| > x\,) \le e^{-x^2/2}$ for $x > 0$ large enough, so $\mathbf{P}(\,|Z_N| > 2\sqrt{\log N}\,) \le N^{-2}$ for all integers N sufficiently large. The Borel-Cantelli Lemma now shows the existence of a random variable $N_0 : \Omega \to \mathbf{N}$ such that $|Z_n(\omega)| \le 2\sqrt{\log n}$, $\forall\, n \ge N_0(\omega)$ holds for a.e. $\omega \in \Omega$. In terms of the original sequence,

$$|Z_{ij}(\omega)| = \left|Z_{N(i,j)}(\omega)\right| \le 2\sqrt{\log N(i,j)} \le 2\sqrt{(i+1)\,\log 2}\,, \quad \text{for } \; i \ge I(\omega)\,,\; j = 1,\cdots,2^i$$

establishing the first estimate.

For the second, note that with $i \in \mathbf{N}$ fixed, the *Schauder functions* $S_{ij}(t) = \int_0^t h_{ij}(u)\,du$ are "narrow tents" over disjoint intervals, of length 2^{-i} and height $2^{-i} \times 2^{i/2}$. Therefore, we have $\sum_j S_{ij}(t) \le 2^{-i/2}$ as well as

$$\sum_j |S_{ij}(t) - S_{ij}(s)| \le |t-s|\cdot 2^{i/2}\,, \tag{9.13}$$

and thus

$$\sup_{0\le t\le 1}\left|\sum_{j=1}^{2^i} Z_{ij}(\omega)\cdot S_{ij}(t)\right| \le \max_{1\le j\le 2^i}|Z_{ij}(\omega)|\cdot\sum_{j=1}^{2^i} S_{ij}(t) \le M_1(\omega)\cdot\sqrt{i}\,2^{-i/2}\,. \qquad \diamond$$

Proof of Theorem 9.1: Let us write the increment $W_t(\omega) - W_s(\omega)$ in two parts

$$W_t(\omega) - W_s(\omega) = Z_{00}(\omega)\cdot(t-s) + \left(\sum_{i=0}^{L} + \sum_{i\ge L+1}\right)\sum_{j=1}^{2^i} Z_{ij}(\omega)\cdot\int_s^t h_{ij}(u)\,du$$

where the integer L will be determined shortly below. From Lemma 9.1 we have the upper bounds

$$\Big| \sum_{i=0}^{L} \sum_{j=1}^{2^i} Z_{ij}(\omega) \cdot \int_s^t h_{ij}(u)\, du \Big| \;\le\; |t-s|\, M_1(\omega) \cdot \sum_{i=0}^{L} \sqrt{i}\, 2^{i/2}$$

using (9.13), the upper bound

$$\Big| \sum_{i \ge L+1} \sum_{j=1}^{2^i} Z_{ij}(\omega) \cdot \int_s^t h_{ij}(u)\, du \Big| \;\le\; M_1(\omega) \sum_{i=L+1}^{\infty} \sqrt{i}\, 2^{-i/2} \;,$$

as well as the elementary estimates $\sum_{i=0}^{L} \sqrt{i}\, 2^{i/2} \;\le\; C_1 \sqrt{L}\, 2^{L/2}$, $\sum_{i=L+1}^{\infty} \sqrt{i}\, 2^{-i/2} \;\le\; C_2 \sqrt{L}\, 2^{-L/2}$. We see then that $|t-s| \le \delta < 1$ leads to

$$|W_t(\omega) - W_s(\omega)| \;\le\; \delta \cdot |Z_{00}(\omega)| \;+\; M_1(\omega) \sqrt{L} \left(\delta\, 2^{L/2} + 2^{-L/2} \right) .$$

The two terms in this last summation can be 'balanced' if we take $\delta\, 2^L \sim 1$ or more precisely $L = [\, \log(1/\delta) \,/\, \log 2 \,]$; substituting in (9.8) we get

$$|W_t(\omega) - W_s(\omega)| \;\le\; \delta \cdot |Z_{00}(\omega)| \;+\; \frac{2 M_1(\omega)}{\sqrt{\log 2}} \cdot \sqrt{\delta\, \log(1/\delta)} \,.$$

This leads to (9.12) with $M(\omega) = |Z_{00}(\omega)| + \frac{2}{\sqrt{\log 2}}\, M_1(\omega)$. ◇

The upper bound in (9.12) suggests a much finer result of P. Lévy, which shows that the function $g(\delta) = \sqrt{2\delta\, \log(1/\delta)}$ is the **exact Modulus of Continuity** of the Brownian path:

$$\limsup_{\delta \downarrow 0} \frac{1}{\sqrt{2\delta\, \log(1/\delta)}} \cdot \max_{\substack{0 \le s < t \le 1 \\ t-s \le \delta}} \big| W_t(\omega) - W_s(\omega) \big| \;=\; 1 \;, \quad \text{for a.e. } \omega \in \Omega \,. \tag{9.14}$$

A related result of A. Hinčin describes the long- and short-time behavior of the Brownian path in terms of the so-called **Law of the Iterated Logarithm** as in Theorem 5.4:

$$\limsup_{t \downarrow 0} \frac{W_t(\omega)}{\sqrt{2t\, \log\log(1/t)}} \;=\; 1 \;, \qquad \liminf_{t \downarrow 0} \frac{W_t(\omega)}{\sqrt{2t\, \log\log(1/t)}} \;=\; -1 \;, \tag{9.15.a}$$

$$\limsup_{t \uparrow \infty} \frac{W_t(\omega)}{\sqrt{2t\, \log\log t}} \;=\; 1 \;, \qquad \liminf_{t \uparrow \infty} \frac{W_t(\omega)}{\sqrt{2t\, \log\log t}} \;=\; -1 \;, \tag{9.15.b}$$

for a.e. $\omega \in \Omega$. Both these results can be proved using the Haar function approach of this subsection; see Dobrić & Marano (2003). (Thanks to the invariance of Brownian

Motion under symmetry and time-inversion, there is only one statement to be proved in the equations (9.15), not four.)

A different approach is illustrated in Karatzas & Shreve (1991), pages 112-116; it proceeds along exactly the same lines as the proof of the Law of the Iterated Logarithm for independent standard gaussian random variables. Since this proof uses martingale methods, it is deferred to Chapter 4, Example 4.4.3.

9.8 Exercise: N. Wiener's original construction of Brownian Motion as a random Trigonometric Series. With $Z_0, Z_1, Z_2, \cdots$ independent standard normal random variables, show that the *Random Trigonometric Series*

$$W_t(\omega) \;:=\; \frac{t}{\sqrt{\pi}}\, Z_0(\omega) \,+\, \sum_{n \in \mathbf{N}} \sum_{k=2^{n-1}}^{2^n - 1} \sqrt{\frac{2}{\pi}} \cdot \frac{\sin(kt)}{k} \cdot Z_k(\omega)\,, \qquad 0 \le t \le \pi$$

converges uniformly for a.e. $\omega \in \Omega$, and that $(t, \omega) \mapsto W_t(\omega)$ is Brownian Motion.