CHAPTER 3: ELEMENTS OF HARMONIC ANALYSIS

In this chapter we introduce the notion and basic properties of Fourier transforms for finite measures and for functions in $\mathbf{L}^1(\mathbf{R})$, including the Fourier-Lévy inversion formula. This fundamental result asserts that a probability measure μ on the real line is uniquely determined by knowledge of its "spectrum" (also called "characteristic function") $\int_{\mathbf{R}} e^{i\xi x}\, d\mu(x)$, for all frequencies $\xi \in \mathbf{R}$. Properties of convergence of measures are tied to related properties of pointwise convergence for the corresponding spectra (characteristic functions), leading to a Fourier-analytic proof of the Central Limit Theorem. The basic principles of the resulting *Harmonic Analysis* are then illustrated on the solution of some simple but fundamental Differential Equations, both Ordinary and Partial (Heat Equation, Wave Equation). The solution to the Heat equation is then expressed in terms of the Brownian Motion process.

3.1. FOURIER TRANSFORMS OF MEASURES

We have seen in the previous chapter how the distribution function $F(\cdot) \equiv F_X(\cdot)$ of a random variable X determines the values of integrals $\mathbf{E}[\Phi(X)] = \int \Phi\, dF_X$ for all $\Phi : \mathbf{R} \to \mathbf{R}$, and how it is in turn determined from these integrals corresponding to all such Φ in the class $C_b(\mathbf{R})$ of bounded continuous functions (cf. Exercise 2.1.4). It turns out that $F_X(\cdot)$ is actually determined from knowledge of the "harmonics"

$$\mathbf{E}(e^{i\xi X}) = \int_{-\infty}^{\infty} \cos(\xi x)\, dF(x) + i \int_{-\infty}^{\infty} \sin(\xi x)\, dF(x)\,,$$

for all "frequencies" $\xi \in \mathbf{R}$. These correspond to the collection of bounded, continuous functions $\left\{e^{i\xi\cdot}\right\}_{\xi\in\mathbf{R}} \subset C_b(\mathbf{R})$; we are denoting here by $i = \sqrt{-1}$ the imaginary unit. The resulting function

$$\mathbf{R} \ni \xi \mapsto \phi_X(\xi) = \mathbf{E}(e^{i\xi X}) \in \mathbf{C}$$

is called the "spectrum" or *characteristic function* of X.

To begin our discussion, we start with a related concept, that of the **characteristic function of a probability measure** μ on Borel subsets of $\mathbf{R}$, defined as

$$\phi_\mu(\xi) := \int_{\mathbf{R}} e^{i\xi x}\, d\mu(x)\,, \qquad \xi \in \mathbf{R}. \tag{1.1}$$

The function $\phi_\mu : \mathbf{R} \to \mathbf{C}$ is then uniformly continuous, and satisfies

$$|\phi_\mu(\xi)| \le \phi_\mu(0) = 1\,, \qquad \phi_\mu(-\xi) = \overline{\phi_\mu(\xi)}\,.$$

• Suppose that μ is absolutely continuous with respect to Lebesgue measure λ, that is, of the form $\mu(A) = \int_A f(u)\,du$, $A \in \mathcal{B}(\mathbf{R})$ for some measurable function $f : \mathbf{R} \to [0,\infty)$ with $\int_{-\infty}^{\infty} f(u)\,du = 1$. Then $\phi_\mu(\cdot)$ coincides with the **Fourier transform** of the probability density function $f(\cdot)$ as in Exercise 1.6.3, namely

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{i\xi u} f(u)\,du\,.$$

• If μ is a purely discrete measure on $\mathbf{N}_0$ (as in Definition 2.1.1(ii)), then $\phi_\mu(\cdot)$ coincides with the **Fourier series** $\widehat{p}(\xi) = \sum_{n\in\mathbf{N}_0} e^{i\xi n} p(n)$ of the probability mass function $p(\cdot)$.

The main property of the function $\phi_\mu(\cdot)$ is that it determines, or "characterizes", the measure μ, whence the terminology "characteristic function". The following fundamental result shows how to reconstruct a measure μ on the real line, from the spectrum of all its harmonics.

1.1 THEOREM : FOURIER-LÉVY INVERSION FORMULA. *For any real numbers $x_1 < x_2$, we have*

$$\underbrace{\mu((x_1,x_2)) + \frac{1}{2}\Big[\mu(\{x_1\}) + \mu(\{x_2\})\Big] = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-i\xi x_1} - e^{-i\xi x_2}}{i\xi}\,\phi_\mu(\xi)\,d\xi\,.}_{} \quad (1.2)$$

A crucial observation here, is that the integrand on the right-hand side of (1.2) is only of the order $O(1/\xi)$ in ξ, thus not integrable on the whole real line. The process of taking the "principal value" of this integral, i.e., of truncation to the interval $(-T,T)$ followed by taking the limit as $T \to \infty$, can be viewed as a way of "regulating" the integral over the entire real line; this feature is common to most proofs of Fourier inversion formulae.

Proof: We start by evaluating the integral

$$\begin{aligned}\frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-i\xi x_1} - e^{-i\xi x_2}}{i\xi}\,\phi_\mu(\xi)\,d\xi &= \int_{-T}^{T} \frac{e^{-i\xi x_1} - e^{-i\xi x_2}}{2\pi i\xi}\left(\int_{\mathbf{R}} e^{i\xi x} d\mu(x)\right) d\xi \\ &= \int_{\mathbf{R}} I_T(x; x_1, x_2)\,d\mu(x)\,. \end{aligned} \quad (1.3)$$

Here we have set

$$\begin{aligned} I_T(x; x_1, x_2) &:= \int_{-T}^{T} \frac{e^{i\xi(x-x_1)} - e^{i\xi(x-x_2)}}{2\pi i\xi}\,d\xi \\ &= \frac{1}{\pi}\int_0^T \frac{\sin\xi(x-x_1)}{\xi}\,d\xi - \frac{1}{\pi}\int_0^T \frac{\sin\xi(x-x_2)}{\xi}\,d\xi \end{aligned} \quad (1.4)$$

and applied Fubini's Theorem. This is justified, since

$$\Big|\frac{e^{i\xi(x-x_1)}-e^{i\xi(x-x_2)}}{i\xi}\Big| = \Big|\int_{x-x_2}^{x-x_1} e^{i\xi u}du\Big| \le x_2-x_1 \tag{1.5}$$

and $\int_{\mathbf{R}} d\mu(x) \int_{-T}^{T}(x_2-x_1)d\xi = 2T(x_2-x_1) < \infty$. The limits of integrals of the form (1.3) are easily determined, with the help of the following so-called 'Dirichlet integrals':

$$\begin{aligned} 0 \le \operatorname{sgn}(\alpha)\cdot \int_0^T \frac{\sin(\alpha x)}{x}\,dx \;&\le\; \int_0^\pi \frac{\sin x}{x}\,dx \\ \lim_{T\to\infty} \int_0^T \frac{\sin(\alpha x)}{x}\,dx \;&=\; \frac{\pi}{2}\cdot\operatorname{sgn}(\alpha) \end{aligned} \tag{1.6}$$

$$\lim_{T\to\infty} \int_0^T \frac{1-\cos(\alpha x)}{x}\,dx \;=\; \frac{\pi}{2}\,|\alpha|$$

with the convention $\operatorname{sgn}(a) = 1, 0, -1$ for $a > 0,\ a = 0,\ a < 0$, respectively. It follows that the quantity $I_T(x; x_1, x_2)$ satisfies $|I_T(x; x_1, x_2)| \le \frac{2}{\pi}\int_0^\pi \frac{\sin x}{x}\,dx$ uniformly in T; thus, as $T\to\infty$, it converges to zero when $x < x_1$ or $x > x_2$; to 1, when $x_1 < x < x_2$; and to $(1/2)$, when $x = x_1$ or $x = x_2$. We can apply now the Lebesgue Dominated Convergence Theorem, and find that the limit of the right-hand side of (1.3) as $T\to\infty$, is exactly $\mu((x_1, x_2)) + \frac{1}{2}\big[\mu(\{x_1\}) + \mu(\{x_2\})\big]$. $\diamondsuit$

Let us list some immediate consequences of the Fourier-Lévy inversion formula.

• In terms of the probability distribution function $F(x) \equiv \mu((-\infty, x])$, $x \in \mathbf{R}$ induced on $\mathcal{B}(\mathbf{R})$ by the probability measure μ, we can re-write the Fourier-Lévy inversion formula in the form

$$\frac{1}{2}\Big[F(x_2)+F(x_2-)\Big] - \frac{1}{2}\Big[F(x_1)+F(x_1-)\Big] \;=\; \lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-i\xi x_1}-e^{-i\xi x_2}}{i\xi}\,\phi_\mu(\xi)\,d\xi\,. \tag{1.7}$$

• **The characteristic function $\phi_\mu(\cdot)$ determines the probability distribution function $F(\cdot)$.** Indeed, for every continuity point x of $F(\cdot)$, we have

$$F(x) = \lim_{x_1\to-\infty}\left(\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-i\xi x_1}-e^{-i\xi t x}}{i\xi}\,\phi_\mu(\xi)\,d\xi\right) \tag{1.7$'$}$$

from (1.7). In other words, $F(\cdot)$ is uniquely determined by $\phi_\mu(\cdot)$ at all its points of continuity – thus everywhere on $\mathbf{R}$ via $F(z) = \lim_{x\downarrow z,\, x\notin D_F} F(x)$; this is because $F(\cdot)$ is increasing and right-continuous, so the set D_F of its discontinuities is at most countable.

• Suppose now that $\phi_\mu \in \mathbf{L}^1(\mathbf{R}) \equiv \mathbf{L}^1(\mathbf{R}, \mathcal{B}(\mathbf{R}), \lambda)$, i.e., $\int_{-\infty}^{\infty} |\phi_\mu(\xi)| d\xi < \infty$ (this is a big assumption; see Exercise 1.7(e) for sufficient conditions). Then *the distribution function* $F(\cdot)$ *is continuously differentiable*, with derivative

$$\underbrace{f(x) \,=\, F'(x) \,=\, \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi x} \phi_\mu(\xi)\, d\xi\,, \quad \forall\, x \in \mathbf{R}\,.} \tag{1.8}$$

Indeed, under the additional hypothesis $\phi_\mu \in \mathbf{L}^1(\mathbf{R})$, the estimate (1.5) and the Lebesgue Dominated Convergence Theorem allow us to replace in (1.2), (1.7) the principal value $\left(\lim_{T\to\infty} \int_{-T}^{T}\right)$ by the Lebesgue integral $\left(\int_{-\infty}^{\infty}\right)$ over the entire real line, namely:

$$\mu((x_1,x_2)) + \frac{1}{2}\Big[\mu(\{x_1\}) + \mu(\{x_2\})\Big] \,=\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - e^{-i\xi(x_2 - x_1)}}{i\xi}\, e^{-i\xi x_1}\, \phi_\mu(\xi)\, d\xi\,.$$

Letting $x_2 \to x_1$ in this expression, and appealing to the Dominated Convergence Theorem once again, we find that $\mu(\{x_1\}) = 0$, so that $F(\cdot)$ is continuous and (1.2) becomes

$$F(x_2) - F(x_1) \,=\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - e^{-i\xi(x_2 - x_1)}}{i\xi}\, e^{-i\xi x_1}\, \phi_\mu(\xi)\, d\xi\,. \tag{1.2$'$}$$

Dividing by $x_2 - x_1$ in this expression, we get

$$\frac{F(x_2) - F(x_1)}{x_2 - x_1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi x_1} \left(\frac{1 - e^{-i\xi(x_2 - x_1)}}{i\xi(x_2 - x_1)} \right) \phi_\mu(\xi)\, d\xi\,.$$

Once again, we let $x_2 \downarrow x_1$ and appeal to the Lebesgue Dominated Convergence Theorem, to conclude that the limit exists and is given by the right-hand side of (1.8) with $x = x_1$.

• The definition of the transform (1.1) also makes sense, if we replace the probability measure μ by a "signed measure" of the form $\mu(A) = \int_A f(x)\, dx = \mu_+(A) - \mu_-(A)$, $A \in \mathcal{B}(\mathbf{R})$ for some $f \in \mathbf{L}^1(\mathbf{R})$ (difference of two finite measures $\mu_\pm(A) = \int_A f^\pm(x)\, dx$ in the notation of (1.1.4)). In this case $\phi_\mu(\cdot)$ is just the **Fourier transform**

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} f(x)\, dx, \quad \xi \in \mathbf{R} \tag{1.9}$$

of the function $f(\cdot)$, as in (1.6.7). It is easy to check that all the previous discussion is still valid, with $d\mu(x)$ replaced formally by $f(x)\, dx$. The resulting function $\widehat{f}(\cdot)$ is bounded and uniformly continuous, with $||\widehat{f}\,||_\infty \le ||f||_1$, but not necessarily in $\mathbf{L}^1(\mathbf{R})$; see Exercise 1.1.(c).

However, if we have $\widehat{f} \in \mathbf{L}^1(\mathbf{R})$, as was assumed in the preceding item, then (1.8) becomes the classical **Fourier inversion formula**

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi x} \widehat{f}(\xi)\, d\xi\,, \quad x \in \mathbf{R}\,. \tag{1.10}$$

In this case it is $f(\cdot)$ that is bounded and uniformly continuous, and $2\pi\,||f||_\infty \le ||\widehat{f}\,||_1$.

- *The characteristic function of the* **convolution**

$$\mu(A) \;\equiv\; (\mu_1 * \mu_2)(A) \;:=\; \int_{\mathbf{R}} \mu_1(A-x)\, d\mu_2(x)\,, \qquad A \in \mathcal{B}(\mathbf{R})$$

of two probability measures μ_1 and μ_2 , *is the* **product** $\phi(\cdot) = \phi_1(\cdot)\cdot\phi(\cdot)$ *of the corresponding characteristic functions.* Indeed, if $X_1,\ X_2$ are two independent random variables with respective distributions μ_1 and μ_2 , then $\mu_1 {*} \mu_2$ is the distribution of $X_1{+}X_2$ (recall Exercise 2.2.4), and its characteristic function is

$$\phi(\xi) = \mathbf{E}\left[e^{i\xi(X_1+X_2)}\right] = \mathbf{E}\left[e^{i\xi X_1}\right]\cdot\mathbf{E}\left[e^{i\xi X_2}\right] = \phi_1(\xi)\cdot\phi_2(\xi)\,, \qquad \forall\ \xi \in \mathbf{R}\,.$$

1.1 Example: The following are the characteristic functions for the absolutely continuous distributions of Example 2.1.1.

- *Exponential:* $\phi(\xi) = \left(1 - i(\xi/\lambda)\right)^{-1}$.
- *Standard Normal:* $\phi(\xi) = e^{-\xi^2/2}$.
- *Normal* $(m\,,\sigma^2)$*:* $\phi(\xi) = e^{\,im\xi - \sigma^2\xi^2/2}$.
- *Uniform on* $[0,h]$*:* $\phi(\xi) = (e^{i\xi h} - 1)/i\xi h$.
- *Uniform on* $[-h,h]$*:* $\phi(\xi) = (\sin(\xi h)/\xi h)$.
- *Double Exponential:* $\phi(\xi) = 1/(1+\xi^2)$.
- *Cauchy:* $\phi(\xi) = e^{-|\xi|}$.
- *Triangular on* $[-1,1]$*:* $\phi(\xi) = 2\,(1-\cos\,\xi)/\xi^2$.
- *Fejér:* $\phi(\xi) = (1-|\xi|)\,\chi_{[-1,1]}(\xi)$.

1.2 Example: The following are the characteristic functions for the purely discrete distributions of Example 2.1.2.

- *Dirac:* $\phi(\xi) = e^{i\xi a}$.
- *Bernoulli:* $\phi(\xi) = p\,e^{i\xi a} + q\,e^{i\xi b}$.
- *Symmetric Bernoulli:* $\phi(\xi) = \cos(\xi b)$.
- *Binomial:* $\phi(\xi) = \left(p\,e^{i\xi} + (1-p)\right)^n$.
- *Poisson:* $\phi(\xi) = \exp\left[\lambda\,(e^{i\xi}-1)\right]$.
- *Geometric:* $\phi(\xi) = p\,e^{i\xi}/\left(1-(1-p)\,e^{i\xi}\right)$.

1.1 Exercise: (a) Verify the computations of the Dirichlet integrals in (1.6).
(b) Verify the computations of Examples 1.1 and 1.2, and justify the "dualities" between the double-exponential and the Cauchy, as well as between the triangular and the Fejér distributions. Observe that the standard normal distribution is "self-dual" in this sense.

(c) Show by example that we can have $|\widehat{f}\,| \notin \mathbf{L}^1(\mathbf{R})$, for suitable $f \in \mathbf{L}^1(\mathbf{R})$, in the notation of (1.9).

1.2 Exercise: (a) Convex combinations of characteristic functions are themselves characteristic functions.
(b) If X, Y are independent random variables, then $\phi_{X+Y}(\cdot) = \phi_X(\cdot)\,\phi_Y(\cdot)$.
(c) If X has a normal distribution with expectation m_1 and variance σ_1^2, and Y has a normal distribution with expectation m_2 and variance σ_2^2, and X and Y are independent, then $X+Y$ has a normal distribution with expectation $m_1 + m_2$ and variance $\sigma_1^2 + \sigma_2^2$.
(d) If $\phi(\cdot)$ is a characteristic function, then so is $|\phi(\cdot)|^2$. (*Hint:* Consider two independent random variables X, Y with common characteristic function $\phi(\cdot)$, and look at $X - Y$; this is the so-called *method of symmetrization.*)

1.3 Exercise: With $\phi(\xi) := (1 - |\xi|)\,\chi_{[-1,1]}(\xi)$ and $\{a_k\}_{k=1}^n \subset (0,\infty)$, $\{\lambda_k\}_{k=1}^n \subset [0,1]$, $n \in \mathbf{N}$ with $\sum_{k=1}^n \lambda_k = 1$ the function $\psi(\xi) = \sum_{k=1}^n \lambda_k\, \phi(\xi/a_k)$, $\xi \in \mathbf{R}$ is a characteristic function. In other words, every even function $\psi : \mathbf{R} \to [0,\infty)$ with $\psi(0) = 1$, whose graph on $[0,\infty)$ is a convex polygon, is a characteristic function.

1.4 Exercise : Positive-Definite Functions. A function $g : \mathbf{R} \to \mathbf{C}$ is called *positive-definite*, if

$$\sum_{j=1}^n \sum_{k=1}^n g(\xi_j - \xi_k)\, z_j\, \bar{z}_k \;\geq\; 0 \qquad \text{holds for any} \quad \{\xi_j\}_{j=1}^n \subset \mathbf{R},\ \{z_j\}_{j=1}^n \subset \mathbf{C},\ n \in \mathbf{N}.$$

(a) Show that every positive-definite function satisfies

$$\sup_{\xi \in \mathbf{R}} |g(\xi + h) - g(\xi)| \;\leq\; 2\sqrt{|1 - g(h)|}\,, \qquad h \in \mathbf{R};$$

in particular, if $g(\cdot)$ is continuous at the origin, then it is uniformly continuous on $\mathbf{R}$.
(b) Show that every characteristic fuction is positive-definite.
(c) *(Bochner-Herglotz)* Every positive-definite function $g : \mathbf{R} \to \mathbf{C}$ with $g(0) = 1$ which is continuous at the origin, is a characteristic function.
(d) *(Pólya)* A function $g : \mathbf{R} \to \mathbf{C}$ with $g(0) = 1$, which is evenly-symmetric, and convex decreasing on $[0,\infty)$, is a characteristic function.
(e) Show that $e^{-|\xi|^\alpha}$, $\xi \in \mathbf{R}$ is a characteristic function for $0 < \alpha \leq 2$.

1.5 Exercise: Show that every characteristic fuction $\phi(\cdot)$ has the properties

$$0 \;\leq\; 1 - \Re(\phi(2\xi)) \;\leq\; 4\,[1 - \Re(\phi(\xi))]\,, \quad \forall\, \xi \in \mathbf{R}$$

$$\int_{-\infty}^{\infty} \frac{1 - \Re(\phi(t))}{t^2}\, dt \;=\; \frac{\pi}{2} \int_{\mathbf{R}} |x|\, d\mu(x)\,.$$

1.6 Exercise: Use the techniques of this section, to derive again the results of Exercises 2.5.1, 2.5.3(a); observe how much more direct these new derivations are.

1.7 Exercise: Let μ, ν be two probability measures on $\mathcal{B}(\mathbf{R})$, and denote their respective characteristic functions by $\phi_\mu(\cdot)\,,\phi_\nu(\cdot)$.
(a) Establish the **Parseval Identity**

$$\int_{\mathbf{R}} e^{-ix\xi}\phi_\mu(\xi)\,d\nu(\xi) \;=\; \int_{\mathbf{R}} \phi_\nu(\xi - x)\,d\mu(\xi)\,, \quad \forall\; x\in\mathbf{R}. \tag{1.11}$$

In particular, for every $a>0\,,\, x\in\mathbf{R}$, show that we have

$$f_a(x;\phi_\mu) \;:=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi-(a\xi)^2/2}\,\phi_\mu(\xi)\,d\xi \;=\; \int_{\mathbf{R}} \frac{e^{-(x-\xi)^2/2a^2}}{a\sqrt{2\pi}}\,d\mu(\xi)\,. \tag{1.12}$$

(*Hint:* Use the Tonelli-Fubini theorems, and recall the characteristic function of the normal distribution $d\nu(x) = \frac{a}{\sqrt{2\pi}}\exp\{-a^2x^2/2\}dx$ with $m=0,\ \sigma=1/a$ from Example 1.1.)
(b) If X is a random variable with distribution μ, and Z is an independent standard normal random variable, then for $a>0$ the distribution of $X{+}aZ$ is absolutely continuous with density $f_a(\,\cdot\,;\phi_\mu)$, namely

$$\mathbf{P}[\,X+aZ\le u\,] \;=\; \int_{-\infty}^{u} f_a(x;\phi_\mu)\,dx\,, \qquad u\in\mathbf{R}. \tag{1.13}$$

(c) Use (1.12), (1.13) to provide another proof of the fact, that μ *is determined completely by its characteristic function* $\phi_\mu(\cdot)$; namely, that the probability distribution function $F(\cdot) := \mu\left((-\infty,\cdot\,]\right)$ is given as

$$F(u) \;=\; \lim_{a\downarrow 0}\int_{-\infty}^{u} f_a(x;\phi_\mu)\,dx\,, \qquad \text{at every continuity point } u \text{ of } F(\cdot)\,. \tag{1.14}$$

(d) Suppose that $\phi_\mu(\cdot)\in\mathbf{L}^1(\lambda)$; then μ is absolutely continuous with respect to Lebesgue measure λ, with density $f = d\mu/d\lambda$ which is bounded and uniformly continuous on $\mathbf{R}$.
(e) Suppose that $\phi_\mu(\xi)\ge 0\,,\ \forall\,\xi\in\mathbf{R}$, that μ is absolutely continuous with respect to Lebesgue measure λ, and that the density $f = d\mu/d\lambda$ is bounded (i.e., $||f||_\infty < \infty$). Then $\phi_\mu(\cdot)\in\mathbf{L}^1(\lambda)$.

1.8 Exercise: The characteristic function $\phi_\mu(\cdot)$ of a probability measure μ is real-valued, if and only if μ is symmetric: $\mu(B) = \mu(-B)\,,\ \forall\; B\in\mathcal{B}$.

1.9 Exercise: Let μ be a probability measure on $\mathcal{B}(\mathbf{R})$, and $\phi_\mu(\cdot)$ its characteristic function as in (1.1).

(a) If for some $m \in \mathbf{N}$ we have $\int_{\mathbf{R}} |x|^m \, d\mu(x) < \infty$, then $\phi_\mu(\cdot)$ has a (uniformly) continuous derivative of order m, given by

$$D^m \phi_\mu(\xi) \equiv \phi_\mu^{(m)}(\xi) \;=\; \int_{\mathbf{R}} (ix)^m \, e^{i\xi x} \, d\mu(x)\,, \qquad \xi \in \mathbf{R}\,. \tag{1.15}$$

In particular, we have then

$$\underbrace{D^m \phi_\mu(0) \;=\; i^m \cdot \int_{\mathbf{R}} x^m \, d\mu(x)\,.}$$

(b) Conversely, if the derivative $D^{2n}\phi_\mu(0)$ of order $m = 2n$ at $\xi = 0$ exists and is finite for some $n \in \mathbf{N}$, then

$$\int_{\mathbf{R}} x^{2n} \, d\mu(x) < \infty \qquad \text{and} \qquad D^{2n}\phi_\mu(0) \;=\; (-1)^n \cdot \int_{\mathbf{R}} x^{2n} \, d\mu(x)\,.$$

1.10 Exercise: For a probability measure μ on Borel subsets of $\mathbf{R}^d$, we define the characteristic function

$$\phi_\mu(\xi) \;:=\; \int_{\mathbf{R}^d} e^{i\langle \xi, x\rangle} \, d\mu(x)\,, \quad \xi \in \mathbf{R}$$

in a manner completely analogous to (1.1) and with the inner-product notation $\langle \xi, x\rangle := \sum_{i=1}^d \xi_i x_i$ in $\mathbf{R}^d$. Most of the results of this chapter have natural analogues in this setting as well. For the distributions of Examples 2.2 and 2.3, verify the computations:

- *Multinomial:* $\phi(\xi) = \left(p_1 \, e^{i\xi_1} + \cdots + p_d \, e^{i\xi_d}\right)^n$.
- *Multivariate Normal:* $\phi(\xi) = \exp\left\{ i\,\langle \xi, m\rangle - \tfrac{1}{2}\langle \xi - m, \, \Sigma(\xi - m)\rangle \right\}$.

1.11 Exercise: Suppose that f and g are real-valued functions in $\mathbf{L}^1(\mathbf{R})$.
If in addition $\widehat{f} \in \mathbf{L}^2(\mathbf{R})$ and $\widehat{g} \in \mathbf{L}^2(\mathbf{R})$, then the **Plancherel Identity**

$$||\widehat{f}\,\overline{\widehat{g}}\,||_1 \;=\; 2\pi \;||f\,g||_1 \tag{1.16}$$

holds. In fact, we have $f \in \mathbf{L}^2(\mathbf{R}) \;\Leftrightarrow\; \widehat{f} \in \mathbf{L}^2(\mathbf{R})$, and in this case the Plancherel Identity becomes

$$||\widehat{f}\,||_2 \;=\; \sqrt{2\pi}\;||f\,||_2\;. \tag{1.17}$$

A: FOURIER TRANSFORMS OF SQUARE-INTEGRABLE FUNCTIONS *

The purpose of this subsection is to discuss the construction and basic properties of the Fourier Transform for *complex-valued functions* $f : \mathbf{R} \to \mathbf{C}$ in the Hilbert space $\mathbf{L}^2(\mathbf{R})$ with

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,dx\,, \qquad ||f||_2 = \left(\int_{-\infty}^{\infty} |f(x)|^2\,dx \right)^{1/2} ; \qquad f\,,\,g \in \mathbf{L}^2(\mathbf{R})\,.$$

The results will not be used in the remainder of this chapter, so the subsection can be skipped or skimmed on first reading.

How then are we to define the Fourier transform

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} f(x)\,dx\,, \quad \xi \in \mathbf{R} \tag{1.9}$$

of a function $f : \mathbf{R} \to \mathbf{C}$ *in* $\mathbf{L}^2(\mathbf{R})$ *?* There is no problem with doing this if, in addition, f belongs to the space $\mathbf{L}^1(\mathbf{R})$; then $\widehat{f} : \mathbf{R} \to \mathbf{C}$ is well-defined, uniformly continuous, and bounded with $||\widehat{f}||_\infty \le ||f||_1 < \infty$, as we have already seen.

Furthermore, *if it happens that* $\widehat{f}$ *is itself integrable,* i.e., $|\widehat{f}| \in \mathbf{L}^1(\mathbf{R})$, then we also have the Fourier Inversion formula (1.10), now written in the form

$$f = \big(\widehat{f}\,\big)^{\vee}, \qquad \text{where} \qquad g^{\vee}(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi x} g(x)\,dx\,, \quad \text{for} \quad g \in \mathbf{L}^1(\mathbf{R})\,; \tag{1.18}$$

in this case f itself is uniformly continuous and bounded, with $2\pi ||f||_\infty \le ||\widehat{f}\,||_1 < \infty$, as we have seen.

However, it is not immediately clear how to use this information to interpret (1.9), let alone the inversion formula (1.10), if we only know that $f \in \mathbf{L}^2(\mathbf{R})$.

None of these problems exists if we take f in the class $C_{\downarrow}^{\infty}(\mathbf{R})$ of infinitely differentiable, rapidly decreasing functions of Definition 1.6.1. In view of the fact that this space is dense in $\mathbf{L}^2(\mathbf{R})$ (Exercise 1.6.9), this may in fact be a good place to start.

To make headway with this idea, let us assume that $f \in C_{\downarrow}^{\infty}(\mathbf{R})$; then the Fourier Transform $\widehat{f}$ is well-defined by (1.9), and it is checked readily that the analogue

$$D^k \widehat{f} = i^k \cdot \widehat{h_k\,f}\,, \quad \forall\; k \in \mathbf{N}_0 \tag{1.19}$$

of (1.15) holds, where we have set $h_k(x) := x^k$. Similarly, integration-by-parts in (1.9) gives

$$\widehat{D^m f} = -i^m \cdot h_m\,\widehat{f}\,, \quad \forall\; m \in \mathbf{N}\,. \tag{1.20}$$

Putting these two properties together, we deduce that $\widehat{f} \in C^{\infty}_{\downarrow}(\mathbf{R})$, and verify the *Fourier inversion formula* (1.10) in this case. One can check also the analogue of the **Parseval Identity**

$$\int_{-\infty}^{\infty} e^{-ix\xi} \widehat{f}(\xi) g(\xi)\, d\xi \;=\; \int_{-\infty}^{\infty} \widehat{g}(\xi - x) f(\xi)\, d\xi\,, \quad \forall\; f,\, g \in C^{\infty}_{\downarrow}(\mathbf{R})\,,\; x \in \mathbf{R} \tag{1.21}$$

of (1.11), as well the **Plancherel Identities**

$$||\widehat{f}\,||_2 \;=\; \sqrt{2\pi}\; ||f\,||_2\;, \qquad ||\widehat{f}\;\overline{\widehat{g}}\,||_1 \;=\; 2\pi\; ||f\,\overline{g}\,||_1\;; \qquad f\,,\, g\; \in C^{\infty}_{\downarrow}(\mathbf{R})\,. \tag{1.22}$$

1.12 Exercise: For $f \in C^{\infty}_{\downarrow}(\mathbf{R})$, $g \;\in C^{\infty}_{\downarrow}(\mathbf{R})$, verify that $\widehat{f} \in C^{\infty}_{\downarrow}(\mathbf{R})$, $\widehat{g} \in C^{\infty}_{\downarrow}(\mathbf{R})$ and check the validity of the properties (1.19)-(1.21).

To establish (1.22), consider the functions $g(x) := \overline{f(-x)}$ and $h(x) := \big(f * g\big)(x) \equiv \int_{-\infty}^{\infty} f(y)\, \overline{f(y-x)}\, dy$, both of them in $C^{\infty}_{\downarrow}(\mathbf{R})$. It is seen that $\widehat{g}$ is the complex conjugate of $\widehat{f}$, and $\widehat{h} \;=\; \widehat{f}\,\widehat{g} \;=\; |\widehat{f}\,|^2$, so that the Fourier Inversion Formula yields

$$\begin{aligned}\Big(||f||_2\Big)^2 \;&=\; \int_{-\infty}^{\infty} f(y)\, \overline{f(y)}\, dy \;=\; h(0) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi x} \widehat{h}(\xi)\, d\xi\, \Big|_{x=0} \\ &=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} \big|\widehat{f}(\xi)\big|^2\, d\xi \;=\; \frac{1}{2\pi}\, ||\widehat{f}\,||^2\,,\end{aligned}$$

which is the first formula in (1.22); the second is proved similarly.

We are now in a position to extend the Fourier transform and its properties, from $C^{\infty}_{\downarrow}(\mathbf{R})$ to the Hilbert space $\mathbf{L}^2(\mathbf{R})$. To do this, take an arbitrary $f \in \mathbf{L}^2(\mathbf{R})$ and any sequence $\{f_n\}_{n=1}^{\infty} \subset C^{\infty}_{\downarrow}(\mathbf{R})$ with $||f_n - f\,||_2 \;\to\; 0$ as $n \to \infty$; such a sequence exists, because $C^{\infty}_{\downarrow}(\mathbf{R})$ is dense in $\mathbf{L}^2(\mathbf{R})$ (Exercise 1.6.9). The Fourier transforms $\{\widehat{f}_n\}_{n=1}^{\infty}$ of these functions are also in $C^{\infty}_{\downarrow}(\mathbf{R})$, and from the Plancherel identity (1.22) we see that

$$\frac{1}{\sqrt{2\pi}} \cdot \big|\big|\widehat{f}_n - \widehat{f}_m\big|\big|_2 \;=\; \big|\big|f_n - f_m\big|\big|_2 \;\le\; \big|\big|f_n - f\,\big|\big|_2 + \big|\big|f_m - f\,\big|\big|_2 \;\longrightarrow\; 0\,, \quad \text{as }\; m\,,\, n \to \infty\,;$$

in other words, $\{\widehat{f}_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the complete space $\mathbf{L}^2(\mathbf{R})$, thus there exists some element $\widehat{f} \in \mathbf{L}^2(\mathbf{R})$ such that: $\big|\big|\widehat{f}_n - \widehat{f}\,\big|\big|_2 \;\longrightarrow\; 0$ as $n \to \infty$. This element is the same (modulo $\lambda-$a.e. equivalence) for any sequence $\{f_n\}_{n=1}^{\infty} \subset C^{\infty}_{\downarrow}(\mathbf{R})$ used to approximate f in the sense of the $\mathbf{L}^2(\mathbf{R})-$norm, so we can define $\widehat{f}$ as **the** *Fourier Transform* of f.

The Plancherel identity follows now immediately, since

$$\big|\big|f\,\big|\big|_2 \;=\; \lim_{n\to\infty} \big|\big|f_n\,\big|\big|_2 \;=\; \frac{1}{\sqrt{2\pi}} \cdot \lim_{n\to\infty} \big|\big|\widehat{f}_n\,\big|\big|_2 \;=\; \frac{1}{\sqrt{2\pi}} \cdot \big|\big|\widehat{f}\,\big|\big|_2\,. \tag{1.23}$$

The inverse map $g \mapsto g^{\vee}$ of (1.18) is extended from $C^{\infty}_{\downarrow}(\mathbf{R})$ to $\mathbf{L}^2(\mathbf{R})$ in exactly the same way, and the *inversion formula* follows:

$$\left(\widehat{f}\,\right)^{\vee} = \left(\lim_{n\to\infty} \widehat{f}_n\,\right)^{\vee} = \lim_{n\to\infty} \left(\widehat{f}\right)^{\vee} = \lim_{n\to\infty} f_n = f\,. \tag{1.24}$$

1.13 Exercise: The Riemann-Lebesgue Lemma. For any $f : \mathbf{R} \to \mathbf{C}$ in $L^1(\mathbf{R})$, show that

$$\lim_{|\xi|\to\infty} |\widehat{f}(\xi)| = 0\,.$$

(*Hint:* Establish this property first for $f \in C^{\infty}_{\downarrow}(\mathbf{R})$; then recall Exercise 1.6.9.)

1.14 Exercise: The Heisenberg "uncertainty principle".
(a) Suppose that $f : \mathbf{R} \to \mathbf{C}$ belongs to the Schwartz space $C^{\infty}_{\downarrow}(\mathbf{R})$. Show then the *Heisenberg Inequality*

$$\int_{-\infty}^{\infty} \Big(x\,|f(x)|\Big)^2 dx \;\cdot\; \int_{-\infty}^{\infty} \Big(\xi\,|\widehat{f}(\xi)|\Big)^2 d\xi \;\geq\; \frac{\pi}{2}\,(\,||f||_2)^4\,, \tag{1.25}$$

with equality for the Gaussian densities $\varphi_{\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}$, $\sigma^2 > 0$.
(*Hint:* Observe that for such f we have $Df \equiv f' \in \mathbf{L}^k(\mathbf{R})$ for every $k \in \mathbf{N}$, and $\widehat{Df}(\xi) = -i\xi\widehat{f}(\xi)$ thanks to integration by parts; then use the Plancherel identity).
(b) Argue that (1.25) holds, in fact, for every $f \in \mathbf{L}^2(\mathbf{R})$.