3.3. PROOF OF THE CENTRAL LIMIT THEOREM

We have now the tools to provide a proof for the Central Limit Theorem 2.5.1. It suffices to deal with the case $m = 0$ (otherwise, we can just replace each X_j by $X_j - m$). We begin by showing that the characteristic function of $\frac{S_n}{\sigma\sqrt{n}}$ converges to the function $e^{-\frac{1}{2}t^2}$, $t \in \mathbf{R}$, as $n \to \infty$. In fact, since the variables $X, X_1, X_2, \cdots$ are independent and have the same distribution, we have

$$\mathbf{E}\left(e^{i\xi\frac{S_n}{\sigma\sqrt{n}}}\right) = \int_{\mathbf{R}^n} e^{i\xi\frac{x_1+\cdots+x_n}{\sigma\sqrt{n}}}\, d\mu_{X_1,\cdots,X_n}(x_1,\cdots,x_n) = \int_{\mathbf{R}^n} e^{i\xi\frac{x_1+\cdots+x_n}{\sigma\sqrt{n}}}\, d\mu(x_1)\cdots d\mu(x_n)$$
$$= \left(\int_{\mathbf{R}} e^{i\xi\frac{x}{\sigma\sqrt{n}}}\, d\mu(x)\right)^n = \left(\mathbf{E}(e^{i\xi\frac{X}{\sigma\sqrt{n}}})\right)^n .$$

To determine the asymptotics of the right-hand side as $n \to \infty$, we use the following version

$$f(x) = f(0) + xf'(0) + \frac{1}{2}x^2 f''(0) + x^2 \int_0^1 (1-u)[f''(ux) - f''(0)]du \tag{3.1}$$

of Taylor's formula, valid for any C^2-function $f(\cdot)$, in particular for $f(x) = e^{ix}$. Setting $x = \frac{\xi X}{\sigma\sqrt{n}}$ and taking expectations, we find

$$\mathbf{E}\left(e^{i\xi\frac{X}{\sigma\sqrt{n}}}\right) = 1 - \frac{\xi^2}{2n} + \frac{\xi^2}{n\,\sigma^2}\cdot \mathbf{E}\left(X^2 \int_0^1 (1-u)\left\{1 - e^{i\frac{u\xi X}{\sigma\sqrt{n}}}\right\} du\right) .$$

Since

$$\left|(1-u)\left\{1 - e^{i\frac{u\xi X}{\sigma\sqrt{n}}}\right\}\right| \le 2 \qquad \text{and} \qquad \mathbf{E}(X^2) < \infty$$

the Lebesgue Dominated Convergence Theorem shows that, for each fixed $\xi \in \mathbf{R}$, we have

$$\mathbf{E}\left(X^2 \cdot \int_0^1 \left\{1 - e^{i\frac{u\xi X}{\sigma\sqrt{n}}}\right\}(1-u)\, du\right) \longrightarrow 0, \qquad \text{thus also} \qquad \mathbf{E}(e^{i\xi\frac{X}{\sigma\sqrt{n}}}) = 1 - \frac{\xi^2 + b_n}{2n},$$

where $b_n \to 0$ as $n \to \infty$; it follows readily that $\mathbf{E}\left(e^{i\xi\frac{S_n}{\sigma\sqrt{n}}}\right) = \left(\mathbf{E}(e^{i\xi\frac{X}{\sigma\sqrt{n}}})\right)^n \to e^{-\frac{1}{2}\xi^2}$. The function $\xi \mapsto e^{-\frac{1}{2}\xi^2}$ is continuous at the origin, and is the characteristic function of the standard normal probability distribution function $\Phi(\cdot)$ of (2.2.7); recall Exercise 1.1. In view of Theorem 2.1, the distribution function of $\frac{S_n}{\sigma\sqrt{n}}$ converges pointwise to $\Phi(\cdot)$. This is indeed the conclusion of the Central Limit Theorem.

3.1 Exercise: Suppose that X, Y are independent random variables with common distribution μ, zero-expectation $\int_{\mathbf{R}} x\, d\mu(x) = 0$ and unit-variance $\int_{\mathbf{R}} x^2\, d\mu(x) = 1$. If $X - Y$, $X + Y$ are independent, then μ is standard normal. (*Hint:* Observe that $\phi_\mu(2\xi) = (\phi_\mu(\xi))^4$, $\forall\, \xi \in \mathbf{R}$, and then iterate.)

3.2 Exercise: In the context of Exercise 2.5.4, and in the case of normal F, show that the random variables $\mathcal{S}_n^2$ and $\overline{X}_n$ are independent.

3.3 Exercise: (a) Show that *the atoms of a measure μ can be recovered from the spectrum* $\phi_\mu(\cdot)$, in the sense

$$\mu(\{x\}) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\xi x} \phi_\mu(\xi)\, d\xi\,, \qquad \forall\, x \in \mathbf{R}.$$

(b) Show that *the total energy in the atoms* equals *the asymptotic energy-per-unit-frequency* in the spectrum, to wit:

$$\sum_{x\in\mathbf{R}} (\mu(\{x\}))^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\phi_\mu(\xi)|^2\, d\xi\,.$$

(*Hint:* Use part (a), and symmetrization.)

3.4 Exercise: Prove the Weak Law of Large Numbers of Theorem 2.3.1, using characteristic functions along with Exercise 2.4.1.

3.5 Exercise: If X_λ has a Poisson distribution with parameter λ, use characteristic functions to show that $(X_\lambda - \lambda)/\sqrt{\lambda}$ converges in distribution to the standard normal.

3.6 Exercise: Let $\{c_n\}_{n\in\mathbf{N}}$ be a sequence of real numbers, such that $\{e^{\,i\xi c_n}\}_{n\in\mathbf{N}}$ converges in $\mathbf{C}$ for every ξ in a set of real numbers with positive measure. Show that $\lim_{n\to\infty} c_n$ exists in $\mathbf{R}$.

3.7 Exercise: Suppose that $\varphi(\cdot)$ is a characteristic function, that $\{b_n\}_{n\in\mathbf{N}}$ is a sequence of positive numbers, and that $\{\varphi(\xi b_n)\}_{n\in\mathbf{N}}$ converges to $\varrho(\xi)$ for all $\xi \in \mathbf{R}$, where $\varrho(\cdot)$ is a characteristic function. Show then that $\lim_{n\to\infty} b_n$ exists in $(0, \infty)$.