3.4. APPLICATIONS TO DIFFERENTIAL EQUATIONS

As our second illustration of Fourier-analytic techniques, this time in the field of Differential Equations, let us consider solving the second-order *Ordinary Differential Equation*

$$u''(x) - u(x) + f(x) = 0\,, \quad x \in \mathbf{R} \tag{4.1}$$

for a function $u \in C^2(\mathbf{R}) \cap \mathbf{L}^1(\mathbf{R})$, where $f : \mathbf{R} \to \mathbf{R}$ is a given function in $C(\mathbf{R}) \cap \mathbf{L}^1(\mathbf{R})$. We impose the requirement $u \in \mathbf{L}^1(\mathbf{R})$ in lieu of "boundary conditions", as we are looking to solve the equation (4.1) on *all of* $\mathbf{R}$.

The solution can be guessed easily, by looking at the Fourier transform $\widehat{u}(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} u(x)\, dx$ of $u(\cdot)$, as well as at those of $u'(\cdot)$, $u''(\cdot)$; "integrating-by-parts" heuristically, and assuming boldly that $u(\pm\infty) = u'(\pm\infty) = 0$, we obtain

$$\widehat{(u')}(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} u'(x)\, dx \;=\; -i\xi \int_{-\infty}^{\infty} e^{i\xi x} u(x)\, dx \;=\; -i\xi\, \widehat{u}(\xi)$$

and $\widehat{(u'')}(\xi) = -i\xi\, \widehat{(u')}(\xi) = -\xi^2 \widehat{u}(\xi)$. But now in light of these computations, and taking Fourier transforms of all terms in (4.1), we obtain from this equation $\widehat{u}(\xi) - \widehat{f}(\xi) = -\xi^2\, \widehat{u}(\xi)$, which implies

$$\widehat{u}(\xi) = \frac{1}{1+\xi^2} \cdot \widehat{f}(\xi)\,, \qquad \xi \in \mathbf{R}\,.$$

But we know from Example 1.1 that $1/(1+\xi^2)$ is the Fourier transform of the double-exponential probability density function $(1/2)\, e^{-|x|}$, and Exercise 1.6.3 suggests that the solution of (4.1) should be the convolution

$$\underbrace{u(x) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-|x-y|} f(y)\, dy\,, \quad x \in \mathbf{R}} \tag{4.2}$$

of the double-exponential density with the given function f. This heuristic approach can be made rigorous, as illustrated in the following two exercises.

4.1 Exercise: Show by direct computation that the function of (4.2)
(i) is of class $C^2(\mathbf{R})$ and satisfies the equation (4.1);
(ii) is in $\mathbf{L}^1(\mathbf{R})$, in fact $||u||_1 \le ||f||_1$.

4.2 Exercise: Show that the function of (4.2) is the only solution of (4.1) in the class $C^2(\mathbf{R}) \cap \mathbf{L}^1(\mathbf{R})$. (*Hint:* Consider any solution v in this class; then $v'' = v - f \in \mathbf{L}^1(\mathbf{R})$, which implies that $\widehat{(v'')}(\xi)$ is well-defined. Argue that $\lim_{|x|\to\infty} v'(x) = 0$ and $\lim_{|x|\to\infty} \big(v(x)\, e^{i\xi x}\big) = 0$, to justify the computation $\widehat{(v'')}(\xi) = -\xi^2\, \widehat{v}(\xi)$. Then use the Fourier inversion results, to conclude $v \equiv u$.)

A: THE WAVE EQUATION ON THE REAL LINE

Our next illustration of Fourier-analytic techniques concerns the **Partial Differential Equation of Wave-Motion**

$$\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{\partial^2 u}{\partial x^2} \;; \qquad t > 0\,,\ x \in \mathbf{R}\,, \tag{4.3}$$

which we shall seek to solve for a $C^{2,2}-$function $u : [0,\infty) \times \mathbf{R} \to \mathbf{R}$ of two arguments (t,x) ("time" and "displacement", respectively) subject to the initial conditions

$$\begin{aligned} &\lim_{t\downarrow 0,\, y\to x} u(t,y) = f(x) \;; \quad x \in \mathbf{R} \qquad \text{(initial displacements)} \\ &\lim_{t\downarrow 0,\, y\to x} \frac{\partial u}{\partial t}(t,y) = g(x) \;; \quad x \in \mathbf{R} \qquad \text{(initial velocities).} \end{aligned} \tag{4.4}$$

Here $c > 0$ is a given real constant, and f, g are given functions in the spaces $C^2_{\downarrow}(\mathbf{R})$ and $C^1_{\downarrow}(\mathbf{R})$, respectively, of Definition 1.6.1. When these functions have compact support, we think of (4.3), (4.4) as describing the wave-motion due to a *localized* initial disturbance in displacements and velocity.

The equation (4.4) can be derived from *Newton's Second Law of Motion,* as follows: consider a small piece AB of the string, stretching at time t as $\{u(t,x),\ a \le x \le b\}$ between the points $x_\ell = a$ and $x_r = b$. The force acting at any position $x \in (a,b)$ comes from the internal tension of the string, which is nearly constant for small oscillations, and acts *along* the string; thus, its vertical component is proportional to the sine

$$\sin(\vartheta) \;=\; \frac{D}{\sqrt{1+D^2}}\,, \qquad D \;:=\; \frac{\partial u}{\partial x}(t,x)$$

of the angle ϑ of inclination. Thus, if $D = \tan(\vartheta)$ is small, the *net* force acting on the piece AB of this string is approximately

$$f \;\simeq\; \text{const} \cdot \left[\frac{\partial u}{\partial x}(t,b) - \frac{\partial u}{\partial x}(t,a) \right].$$

By Newton's Second Law of Motion,

$$f \;=\; \text{mass} \cdot \text{acceleration} \;\simeq\; \text{const.}\,(b-a) \cdot \frac{\partial^2 u}{\partial t^2}(t,x)\,.$$

Equating the two expressions we see that for $b-a$ small:

$$\frac{\partial^2 u}{\partial t^2}(t,x) \;\simeq\; \text{const} \cdot \frac{1}{b-a} \left[\frac{\partial u}{\partial x}(t,b) - \frac{\partial u}{\partial x}(t,a) \right] \;\simeq\; \text{const} \cdot \frac{\partial^2 u}{\partial x^2}(t,x)\,,$$

as postulated by the equation (4.4).

To solve the initial-value problem of (4.3)-(4.4) for the Wave Equation, we look again at the Fourier transform

$$\widehat{u}(t,\xi) := \int_{-\infty}^{\infty} e^{i\xi x} u(t,x)\, dx$$

of $u(t,\cdot)$ for each fixed $t \geq 0$. Proceeding heuristically, by differentiation and integration-by-parts, we obtain much as before

$$\frac{\partial^2 \widehat{u}}{\partial t^2}(t,\xi) \;=\; \int_{-\infty}^{\infty} e^{i\xi x} \frac{\partial^2 u}{\partial t^2}(t,x)\, dx \;=\; c^2 \int_{-\infty}^{\infty} e^{i\xi x} \frac{\partial^2 u}{\partial x^2}(t,x)\, dx \;=\; \cdots \;=\; -(c\xi)^2\, \widehat{u}(t,\xi)\,.$$

The advantage here, is that the resulting equation

$$\frac{\partial^2}{\partial t^2}\, \widehat{u}(t,\xi) \;=\; -(c\xi)^2\, \widehat{u}(t,\xi)$$

is, for fixed $\xi \in \mathbf{R}$, a second-order *ordinary* differential equation in the temporal variable, which can be solved very easily subject to the initial conditions

$$\widehat{u}(0,\xi) \;=\; \widehat{f}(\xi)\,, \qquad \frac{\partial\, \widehat{u}}{\partial t}(0,\xi) \;=\; \widehat{g}(\xi)$$

for each fixed $\xi \in \mathbf{R}$; the solution is

$$\widehat{u}(t,\xi) \;=\; \cos(ct\xi)\cdot \widehat{f}(\xi) + \frac{\sin(ct\xi)}{ct\xi}\cdot \widehat{g}(\xi)\, t\,; \quad t > 0\,. \tag{4.5}$$

But now let us change our point of view, and look at this expression as a fucntion of $\xi \in \mathbf{R}$ for each for fixed $t > 0$; we know from Examples 1.1 and 1.2 that the functions $\xi \mapsto \cos(ct\xi)$, $\xi \mapsto \big(\sin(ct\xi)/ct\xi\big)$ are the Fourier transforms of the symmetric Bernoulli distribution $(\delta_{ct} + \delta_{-ct})/2$ and of the uniform distribution with density $(1/2ct)\, \chi_{[-ct,ct]}$, respectively. Thus, in order to invert (4.5), we can take the convolution of the first of these distributions with $f(\cdot)$ and of the second with $t\, g(\cdot)$, and arrive at the famous **D'Alembert formula**

$$\underbrace{u(t,x) \;=\; \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\, dy\,; \quad t \in [0,\infty),\ x \in \mathbf{R}\,.}_{} \tag{4.6}$$

It is not hard to verify that this function solves the initial-value problem (4.3)-(4.4) for the Wave Equation. The careful reader may have noticed already, in the expression of (4.6), the familiar wedge-like shape of wavefront propagation that can be observed, for instance, in the wake of a boat travelling at constant speed in a calm lake: the initial disturbance propagates at the *constant speed* $c > 0$.

B: THE HEAT EQUATION ON THE REAL LINE

We illustrate further the applicability of Fourier transforms, by solving the **Cauchy** *(initial-value)* **problem**

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \;\; ; \quad t > 0\,,\;\; x \in \mathbf{R} \tag{4.7}$$

$$\lim_{t \downarrow 0} u(t,x) \;=\; f(x) \;\; ; \quad x \in \mathbf{R}\,, \tag{4.8}$$

for the **Partial Differential Equation of Heat Transfer** on an infinite linear rod. Here $f : \mathbf{R} \to \mathbf{R}$ is a given function, whose regularity properties will be specified below; it plays the rôle of "initial temperature profile", in the sense that $u(0,x) \equiv f(x)$ is the temperature at $t = 0$ at the position $x \in \mathbf{R}$ on the infinite rod. The problem is to determine the tempereture-profile $u(t,x),\; x \in \mathbf{R}$ along the rod, at all subsequent times $t > 0$.

The equation (4.7) can be derived from *Newton's Law of cooling*, as follows: according to this law, the heat-flux across a certain point x (from left to right) is proportional to the temperature gradient at x, so the total such flux during a short time-interval $(t, t+\delta)$ is approximately $-\kappa\delta\,(\partial u/\partial x)$. Here the constant $\kappa > 0$ is the heat-conductivity of the material, and the sign reflects the fact that heat flows form hot places to cool. Therefore, the *net amount* of heat flowing *out* of a small neighborhood $I = (x-h, x+h)$ during the short time-interval $(t, t+\delta)$ is approximately

$$-\kappa\delta\rho\left[\frac{\partial u}{\partial x}(t, x+h) - \frac{\partial u}{\partial x}(t, x-h)\right],$$

where $\rho > 0$ is the density of the material. However, this same net-amount can also be computed approximately as

$$-\,2h\,c\,\left[u(t+\delta, x) - u(t,x)\right],$$

where the "specific-heat" $c > 0$ of the material multiplies the average decrease in temperature over the neighborhood I during the short time-interval $(t, t+\delta)$. Equating these two expressions, dividing by $2h\delta$, and then letting both h and δ decrease to zero, we arrive at the equation $\frac{\partial u}{\partial t} = (\sigma/2)\,\frac{\partial^2 u}{\partial x^2}$ with $\sigma = \kappa\rho/c$; the equation (4.7) corresponds then to the normalization $\sigma = 1$.

The problem of solving (4.7), (4.8) can be reduced formally to another, much simpler initial-value problem, for the Fourier transform

$$\widehat{u}(t,\xi) := \int_{-\infty}^{\infty} e^{i\xi x} u(t,x)\, dx$$

of the function $x \mapsto u(t,x)$, if it is assumed that the initial datum $f(\cdot)$ belongs to the space $\mathbf{L}^1(\mathbf{R}) \cap \mathbf{L}^2(\mathbf{R})$. For each fixed $\xi \in \mathbf{R}$, formal differentiation under the integral sign, followed by integrations by parts, yields then

$$\frac{\partial \widehat{u}}{\partial t}(t,\xi) \;=\; \int_{-\infty}^{\infty} e^{i\xi x}\, \frac{\partial u}{\partial t}(t,x)\, dx \;=\; \int_{-\infty}^{\infty} e^{i\xi x}\, \frac{1}{2}\, \frac{\partial^2 u}{\partial x^2}(t,x)\, dx \;=\; \cdots \;=\; -\frac{1}{2}\, \xi^2\, \widehat{u}(t,\xi)$$

a first-order *ordinary* differential equation for the function $t \mapsto \widehat{u}(t,\xi)$, subject to the initial condition $\widehat{u}(0,\xi) = \hat{f}(\xi) := \int_{-\infty}^{\infty} e^{i\xi x} f(x) dx$. The solution is seen to be

$$\widehat{u}(t,\xi) \;=\; \widehat{f}(\xi) \cdot e^{-t\xi^2/2}\,; \qquad t \geq 0\,. \tag{4.9}$$

• Now we change our point of view, and look at the expression of (4.9) as a function of $\xi \in \mathbf{R}$, for each fixed $t \geq 0$. From the Plancherel identity (1.17), we deduce that the resulting function $\widehat{u}(t,\cdot)$ is in $\mathbf{L}^2(\mathbf{R})$ for each $t \geq 0$, as well as in $\mathbf{L}^1(\mathbf{R})$ for each $t > 0$ by the Cauchy-Schwartz inequality. The Fourier inversion formula (1.10) allows us then to define $u(t,\cdot)$ for each $t > 0$ as a uniformly continuous function in the space $\mathbf{L}^\infty(\mathbf{R})$, namely

$$u(t,x) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\xi - t\xi^2/2}\, \widehat{f}(\xi)\, d\xi\,, \quad \forall\; x \in \mathbf{R}\,. \tag{4.10}$$

We can then differentiate under the integral sign repeatedly, and deduce that $u(t,\cdot)$ belongs to the space $C^\infty(\mathbf{R})$ and satisfies the heat equation (4.7), for $t > 0$. Furthermore, let us notice that the right-hand side of (4.10) is well-defined also for $t = 0$, and coincides then with the $\mathbf{L}^2(\mathbf{R})-$function

$$x \mapsto f(x) \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\xi x}\, \widehat{f}(\xi)\, d\xi$$

of (1.10). The Plancherel identity (1.17) of Exercise 1.11 shows $u(t,\cdot) \in \mathbf{L}^2(\mathbf{R})$ for all $t \geq 0$, and we deduce that the initial condition is also satisfied, but now in the weak sense

$$\big|\big|u(t,\cdot) - f(\cdot)\big|\big|_2 \longrightarrow 0\,, \qquad \text{as} \;\; t \downarrow 0\,. \tag{4.8$'$}$$

Indeed, by (1.17) and the Lebesgue Dominated Convergence Theorem,

$$2\pi \left(\big|\big|u(t,\cdot) - f(\cdot)\big|\big|_2\right)^2 \;=\; \left(\big|\big|\widehat{u}(t,\cdot) - \widehat{f}(\cdot)\big|\big|_2\right)^2 \;=\; \int_{-\infty}^{\infty} \left|e^{-t\xi^2/2} - 1\right|^2 \left|\widehat{f}(\xi)\right|^2 d\xi \;\longrightarrow\; 0$$

as $t \downarrow 0$, since $\widehat{f}(\cdot) \in \mathbf{L}^2(\mathbf{R})$ (again by Exercise 1.11).

• For an initial datum $f(\cdot)$ in the Schwartz space $C^\infty_\downarrow(\mathbf{R})$ of rapidly decreasing, infinitely differentiable functions (Definition 1.6.1), the function $x \mapsto u(t,x)$ is smooth for $t \geq 0$, and the initial condition reduces to its simple form (4.8). Indeed, $\widehat{f}(\cdot)$ is then also in $C^\infty_\downarrow(\mathbf{R})$; we can differentiate an arbitrary number of times and then let $t \downarrow 0$ in (4.10), since all the resulting integrals are then absolutely convergent.

• To deal with larger classes of initial data $f(\cdot)$, it is more convenient to recast the formula (4.10) formally, with the help of Fubini-Tonelli, in yet another form

$$\begin{aligned} u(t,x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi}\, e^{-t\xi^2/2}\left(\int_{-\infty}^{\infty} e^{iy\xi} f(y)\,dy\right) d\xi \\ &= \int_{-\infty}^{\infty} f(y)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\,i(y-x)\xi \,-\, t\xi^2/2}\, d\xi\right] dy\,. \end{aligned}$$

The inner integral amounts to an easy Gaussian calculation, to wit,

$$\underbrace{u(t,x) = \int_{-\infty}^{\infty} p_t(x,y) f(y)\,dy\,, \quad t>0\,,\ x\in\mathbf{R}\,,} \tag{4.11}$$

where

$$p_t(x,y) := \frac{1}{\sqrt{2\pi t}}\, e^{-(x-y)^2/2t}\,, \quad t>0\,,\ x\in\mathbf{R}\,,\ y\in\mathbf{R} \tag{4.12}$$

is the fundamental Gaussian *("heat")* kernel of (2.9.3).

Alternatively, we may observe that the right-hand side of (4.9) is the product of the Fourier transform of $f(\cdot)$ with that of the Gaussian probability density function $y \mapsto (2\pi t)^{-1/2}\, e^{-y^2/2t}$ (Exercise 1.1), thus identifying $u(t,\,\cdot\,)$ as the **convolution** of these two functions in the manner of (4.11) and (4.12).

• We can now use (4.11) to construct solutions of the initial-value problem of (4.7)-(4.8) for the heat equation, corresponding to more general classes of initial data. For example, assume that the initial temperature profile $f(\cdot)$ is a *uniformly continuous* function on $\mathbf{R}$. Then (4.11) defines a smooth function $u(t,\cdot)$ for $t>0$; the mapping $(t,x)\mapsto u(t,x)$ is actually continuous on $[0,\infty)\times\mathbf{R}$ (that is, all the way down to $t=0$), while its restriction to $t=0$ is $f(\cdot)$. For this, we note that the heat-kernel $p_t(x,y)$ satisfies the following conditions, for each $x\in\mathbf{R}$:

(i) $\int_{-\infty}^{\infty} p_t(x,y)dy = 1$, for each $t>0$.
(ii) $p_t(x,y)>0$, for all $t>0$, $y\in\mathbf{R}$.
(iii) For any $\delta>0$, we have $\int_{\{y\in\mathbf{R}\,:\,|x-y|>\delta\}} p_t(x,y)\,dy \longrightarrow 0$, as $t\downarrow 0$.

The first item follows from the fact that $y\mapsto p_t(x,y)$ is a probability density function (of the normal distribution with expectation x and variance t). The second is also obvious. As for the third item, we note that the change of variables to $z=(x-y)t^{-1/2}$ transforms the integral given there into $(2\pi)^{-1/2}\int_{\{z\in\mathbf{R}\,:\,|z|>\delta t^{-1/2}\}} e^{-|z|^2/2}dz \,=\, 2\left[1-\Phi(\delta/\sqrt{t})\right] \,\rightarrow\, 0$ as $t\downarrow 0$, in the notation of (2.5.1). Let δ be now any positive number, and write

$$\begin{aligned} u(t,x)-f(x) \,&=\, \int_{-\infty}^{\infty}[f(y)-f(x)]\,p_t(x,y)\,dy \\ &=\, \left(\int_{\{y\,:\,|x-y|\le\delta\}} + \int_{\{y\,:\,|x-y|>\delta\}}\right)[f(y)-f(x)]\,p_t(x,y)\,dy\,, \end{aligned}$$

in view of (i). Since $f(\cdot)$ is uniformly continuous, we can make the first integral on the right-hand side of this last expression smaller than any given number $\varepsilon>0$, by choosing $\delta>0$ small enough. In view of (iii), the second integral can also be made smaller than ε, by choosing $t>0$ small enough. This establishes the continuity of $u(\cdot,\cdot)$ on $[0,\infty)\times\mathbf{R}$, namely (4.8); or even

$$\lim_{t\downarrow 0,\,\zeta\rightarrow x} u(t,\zeta) = f(x)\,, \quad \forall\, x\in\mathbf{R}\,. \tag{4.13}$$

4.3 Exercise : General Solution of the Cauchy Problem for the Heat-Equation. Suppose that the initial temperature-profile function $f:\mathbf{R}\rightarrow\mathbf{R}$ is measurable, and satisfies

$$\int_{-\infty}^{\infty} e^{-x^2/2T}\,|f(x)|\,dx \,<\, \infty \tag{4.14}$$

for some $T\in(0,\infty)$. Then the function u of (4.11) is well-defined and of class C^∞ on $(0,T)\times\mathbf{R}$ (on $(0,\infty)\times\mathbf{R}$, if (4.14) holds for *all* $T>0$), and satisfies on this strip the heat equation (4.7). If, furthermore, the function $f(\cdot)$ is continuous, then u satisfies both the heat equation (4.7) and the initial condition (4.13).

4.4 Exercise : Tychonoff's Uniqueness Theorem. Suppose that the functions $u_j(\cdot,\cdot)$, $j=1,2$ are of class $C^{1,2}$ on the strip $(0,T]\times\mathbf{R}$ and satisfy the heat-equation (4.7) there, as well as the conditions $\lim_{t\downarrow 0,\,y\rightarrow x} u_1(t,y) = \lim_{t\downarrow 0,\,y\rightarrow x} u_2(t,y)$ and

$$\sup_{0<t\le T}\left|u_j(t,x)\right| \,\le\, Ke^{ax^2}\,, \quad j=1,2$$

for all $x\in\mathbf{R}$, where $K>0$ and $a>0$ are real constants. Then $u_1(\cdot,\cdot)\equiv u_1(\cdot,\cdot)$ on $(0,T]\times\mathbf{R}$.

4.5 Exercise : Non-negative Solutions of the Heat Equation (*Widder, 1944*)**.**
(i) Suppose that the *nonnegative* function u is defined and of class $C^{1,2}$ on $(0,T)\times\mathbf{R}$ for some $T\in(0,\infty]$, and satisfies on this strip the heat equation (4.7). Then there exists a measure μ on $\mathcal{B}(\mathbf{R})$, such that

$$u(t,x) = \int_{-\infty}^{\infty} p_t(x,y)\,d\mu(y)\,, \quad 0<t<T\,,\; x\in\mathbf{R} \tag{4.15}$$

in the notation of (4.12). Conversely, every function u of the form (4.15) is a nonnegative solution of the heat equation.

(ii) Similarly, let v be a nonnegative function defined and of class $C^{1,2}$ on $(0,\infty)\times\mathbf{R}$, which satisfies on this strip the *backward heat equation*

$$\frac{\partial v}{\partial t} + \frac{1}{2}\,\frac{\partial^2 v}{\partial x^2} = 0\,. \tag{4.7$'$}$$

Then there exists a a measure μ on $\mathcal{B}(\mathbf{R})$, so that

$$u(t,x) = \int_{-\infty}^{\infty} \exp\left(xy - \frac{y^2}{2}\,t\right)\,d\mu(y)\,, \quad 0<t<\infty\,,\ x\in\mathbf{R}\,; \tag{4.15$'$}$$

and conversely, every function v of the form (4.15)$'$ is a nonnegative solution of the backward heat equation.

4.1 Remark: Consider the expression of (4.15) for the measure $\mu=\delta_a$; this amounts to placing a unit of heat at the site $a\in\mathbf{R}$, while leaving the rest of the rod at temperature $f(x)=0\,,\ \ \forall\, x\neq a$. Then (4.15)$'$ becomes

$$v(t,x)\ =\ \frac{1}{\sqrt{2\pi t}}\,e^{-(x-a)^2/2t}\ ,$$

a quantity which is strictly positive for *any* $t>0$ and $x\in\mathbf{R}$. In other words, *heat is transferred across an ideal conductor at infinite speed*. This is in sharp contrast with the qualitative properties of the wave equation, studied in paragraph 3.4.1; as we saw there, an initial wave-disturbance propagates at a constant, *finite* speed.

4.6 Exercise : Neumann Boundary Data, Insulated Heat Flow on a Semi-Infinite Rod. Suppose that the continuous function $f:\mathbf{R}\to\mathbf{R}$ is evenly symmetric on $\mathbf{R}$ and satisfies (4.14) for all $T\in(0,\infty)$; then the function $u(t,\cdot)$ of (4.11), solution of the initial-value problem of (4.7)-(4.8) for the Heat Equation, inherits this even symmetry, and actually solves the *Initial-Boundary Value Problem with Neumann Boundary Condition*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\,\frac{\partial^2 u}{\partial x^2}\ ;\quad t>0\,,\ x>0 \qquad \text{(Heat Equation)}$$

$$\lim_{t\downarrow 0,\, y\to x} u(t,y) = f(x)\ ;\quad x>0 \qquad \text{(Initial Condition)}$$

$$\frac{\partial u}{\partial x}(t,0) = 0\ ;\quad t>0 \qquad \text{(Neumann Boundary Condition).}$$

In other words, in order to determine the heat-flow on a semi-infinite rod, whose endpoint is kept insulated at all times $\left(\frac{\partial u}{\partial x}(t,0)=0 \text{ at } x=0\,,\forall\, t>0\right)$, it suffices to compute the heat-flow on an infinite rod, with the initial temperature-profile extended by *even symmetry* on $(0, -\infty)$. This, and the resulting formula

$$\underbrace{u(t,x) \;=\; \int_0^\infty \left[\, p_t(x,y) + p_t(x,-y) \right]\, f(y)\, dy \;; \qquad t>0\,,\; x \geq 0} \tag{4.16}$$

for the above initial-boundary value problem, constitute *Lord Kelvin's Method of Images* for this problem.

4.7 Exercise : Dirichlet Boundary Data, Heat Flow on a Semi-Infinite Rod with Endpoint "frozen" at all Times. Suppose now that the continuous function $f : \mathbf{R} \to \mathbf{R}$ is *oddly* symmetric on $\mathbf{R}$ and satisfies (4.14) for all $T \in (0,\infty)$; then the function $u(t,\cdot)$ of (4.11) inherits this odd symmetry, and solves the *Initial-Boundary Value Problem with Dirichlet Boundary Condition*

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\;; \quad t>0\,,\; x>0 && \text{(Heat Equation)} \\ \lim_{t\downarrow 0,\, y\to x} u(t,y) &= f(x)\;; \quad x>0 && \text{(Initial Condition)} \\ u(t,0) &= 0\;; \quad t>0 && \text{(Dirichlet Boundary Condition).} \end{aligned}$$

In other words, in order to determine the heat-flow on a semi-infinite rod, whose end-point is kept at a constant temperature of 0 degrees Celsius (freezing) at all times ($u(t,0)=0$ at $x=0\,, \forall\, t>0$), it suffices to compute the heat-flow on an infinite rod, with the initial temperature-profile extended by *odd symmetry* on $(0,-\infty)$. This, and the resulting formula

$$\underbrace{u(t,x) \;=\; \int_0^\infty \left[\, p_t(x,y) - p_t(x,-y) \right]\, f(y)\, dy \;; \qquad t>0\,,\; x \geq 0} \tag{4.17}$$

for the above initial-boundary value problem, constitute *Lord Kelvin's Method of Images* in this case.

4.8 Exercise : Dirichlet Boundary Data; Heat Flow on an initially frozen Semi-Infinite Rod, with gradual warming at the Endpoint. Consider the Initial-Boundary Value Problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\;; \quad t>0\,,\; x>0 && \text{(Heat Equation)} \\ \lim_{t\downarrow 0,\, y\to x} u(t,y) &= 0\;; \quad x>0 && \text{(Initial Condition)} \\ u(t,0) &= g(t)\;; \quad t>0 && \text{(Dirichlet Boundary Condition).} \end{aligned}$$

for some bounded, continuous function $g : [0,\infty) \to [0,\infty)$.

This corresponds to setting the initial temperature at zero throughout a semi-infinite rod, whose end-point $x = 0$ is now assigned a time-varying temperature profile. Show that this problem is solved by the *Abel Tranform*

$$\underbrace{u(t,x) \;=\; \int_0^t \frac{x}{s}\, p_s(x,0)\, g(t-s)\, ds\,; \quad t>0\,,\; x>0\,.} \tag{4.18}$$

4.9 Exercise : Schrödinger Equation. Show that the function

$$V(t,x) \;=\; -\log\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}}\; \exp\left\{ -h(y) - \frac{(x-y)^2}{2t} \right\} dy \right)\,; \quad t>0,\; x\in \mathbf{R}$$

solves the initial-boundary value problem for the *Schrödinger Equation*

$$\frac{\partial V}{\partial t} \;=\; \frac{1}{2}\left[\frac{\partial^2 V}{\partial x^2} - \left(\frac{\partial V}{\partial x} \right)^2 \right] \;\;;\;\; t>0\,,\; x\in\mathbf{R}$$

$$\lim_{t\downarrow 0,\, y\to x} V(t,y) \;=\; h(x)\;;\;\; x\in\mathbf{R}\,,$$

where $h:\mathbf{R}\to\mathbf{R}$ is a given function with $\int_{-\infty}^{\infty} e^{-\left(h(x)+ax^2\right)}\, dx \,<\, \infty\,,\;\; \forall\; a>0\,.$

4.10 Exercise : Burgers Equation. For a given continuous function $\psi:\mathbf{R}\to\mathbf{R}$ and with $\Psi(t,x,y) := (x-y)^2/2t + \int_0^y \psi(\xi)\, d\xi$, the function

$$v(t,x) \;=\; \frac{\int_{-\infty}^{\infty} \left(\frac{x-y}{t}\right)\, e^{-\Psi(t,x,y)}\, dy}{\int_{-\infty}^{\infty} e^{-\Psi(t,x,y)}\, dy}$$

solves the initial-boundary value problem for the *Burgers Equation*

$$\frac{\partial v}{\partial t} \;=\; \frac{1}{2}\,\frac{\partial^2 v}{\partial x^2} - v\left(\frac{\partial v}{\partial x}\right) \;\;;\;\; t>0\,,\; x\in\mathbf{R}$$

$$\lim_{t\downarrow 0,\, y\to x} v(t,y) \;=\; \psi(x)\;;\;\; x\in\mathbf{R}\,.$$

4.11 Exercise: For a given bounded continuous function $\sigma : [0,\infty) \to (0,\infty)$ with $\int_0^\infty \sigma(t)dt = \infty$, consider the non-linear partial differential equation

$$\left(\frac{\partial^2 V}{\partial x^2}\right)\left(\frac{\partial V}{\partial t}\right) = \frac{\sigma(t)}{2}\left(\frac{\partial V}{\partial x}\right)^2 , \quad t > 0\,,\ x > 0\,.$$

Consider solutions $V(t,x) : (0,\infty)\times(0,\infty) \to \mathbf{R}$ of this equation which are of class $C^{1,3}$ and such that the function $V(t,\cdot)$ is strictly increasing, strictly concave with

$$\frac{\partial V}{\partial x}(t,0+) = \infty\,, \qquad \frac{\partial V}{\partial x}(t,\infty) = 0\,.$$

(i) Show that any such solution is of the form $V(t,x) = \inf_{y>0}\big[Q(t,y)+xy\big]$, where

$$Q(t,y) = \int_{(0,\infty)} \frac{1}{1-s}\left(1 - y^{1-s}\, e^{A(t)\,s(1-s)}\right)\nu(ds) + C\ ;$$

here C is a constant, $A(t) = (1/2)\int_0^t \sigma(u)\,du$, and ν a finite measure on $\mathcal{B}\big((0,\infty)\big)$ with Laplace transform which is finite everywhere.

(ii) Let $u : (0,\infty) \to \mathbf{R}$ be a given strictly increasing, strictly concave function of class C^1 and with $u'(0+) = \infty$, $u'(\infty) = 0$. There exists a function $V(t,x) : (0,\infty)\times(0,\infty) \to \mathbf{R}$ as above and with $V(0+,x) = u(x)$ for all $x \in (0,\infty)$, if and only if

$$\big(u'\big)^{-1}(y) = \int_{(0,\infty)} y^{-s}\,\nu(ds)\,, \qquad 0 < y < \infty$$

for some finite measure ν on $\mathcal{B}\big((0,\infty)\big)$ with Laplace transform which is finite everywhere. Determine this measure explicitly, in the case $u(x) = x^\gamma/\gamma$, $x > 0$ for some $\gamma \in (0,1)$.

$\big($*Hint:* Start with a solution V of this form, and look at its convex dual

$$Q(t,y) := \sup_{x>0}\big[V(t,x) - xy\big]\,, \qquad y > 0\,;$$

write down the partial differential equation satisfied by the maximizer

$$I(t,y) = -\frac{\partial Q}{\partial y}(t,y) > 0 \quad \text{defined implicitly via} \quad \frac{\partial V}{\partial x}\big(t, I(t,y)\big) = y\,,$$

then recall the representation $(4.15)'$ of positive solutions to the backwards heat equation. Now retrace the steps.$\big)$

C. THE HEAT EQUATION AND BROWNIAN MOTION

The coincidence of the transition kernel (2.9.3) for the standard Brownian Motion and the fundamental solution (4.12) of the Heat Equation, suggests a deep connection that exists between the probabilistic and the analytical object. We shall point out in this subsection just a few instances of this connection; for a more detailed treatment we send the reader to Chapter 4 in Karatzas & Shreve (1991).

Let us recall the expressions (2.9.2), (2.9.3) for the transition probabilities of standard Brownian Motion. They show that we may cast the expression (4.11) for the solution of the **Cauchy problem** of (4.7), (4.8) **for the Heat Equation** in the suggestive form

$$\begin{aligned} u(t,x) \;=\; \mathbf{E}[f(x+W_t)] \;&=\; \int_{-\infty}^{\infty} \frac{e^{-\xi^2/2t}}{\sqrt{2\pi t}}\, f(x+\xi)\, d\xi \\ &=\; \int_{-\infty}^{\infty} p_t(x-y)\, f(y)\, dy\,, \quad t>0,\; x \in \mathbf{R}\,. \end{aligned} \tag{4.19}$$

In other words: we can compute the temperature at time t at the site x along an infinite rod, by starting a Brownian motion at that site, letting it run backwards it time for t time-units, and then "averaging out" the initial temperature profile $u(t,x) = E[f(x+W_t)]$ over all its possible terminal positions.

- Similarly, the expression

$$\begin{aligned} u(t,x) \;=\; \mathbf{E}[f(|x+W_t|)] \;&=\; \int_{-\infty}^{\infty} \frac{e^{-\xi^2/2t}}{\sqrt{2\pi t}}\, f(|x+\xi|)\, d\xi \\ =\; \int_{-\infty}^{\infty} p_t(x-y)\, f(|y|)\, dy \;&=\; \int_0^{\infty} \big[p_t(x-y)+p_t(x+y)\big]\, f(y)\, dy\,, \quad t>0,\; x>0 \end{aligned} \tag{4.20}$$

provides the solution to the **Initial/BoundaryValue Problem** of Exercise 4.6 **with Neumann boundary data** $\frac{\partial u}{\partial x}(t,0)=0$. In words: to compute the temperature at time t and at site $x>0$ along a semi-infinite rod, start a Brownian motion at that site; let it run backwards it time for t time-units while at the same time *reflecting* it whenever it hits the origin, and then "average out" the initial temperature profile $u(t,x) = E[f(|x+W_t|)]$ as before.

- Now let us look at the **Initial/Boundary Value Problem with Dirichlet boundary data**

$$\frac{\partial u}{\partial t} = \frac{1}{2}\,\frac{\partial^2 u}{\partial x^2}\;;\quad t>0\,,\; x>0$$

$$\lim_{\substack{t\downarrow 0\\ y\to x}} u(t,y) = f(x)\;;\quad x>0$$

$$u(t,0) = g(t)\;;\quad t>0$$

for given bounded, continuous functions $f : [0,\infty) \to \mathbf{R}$ and $g : [0,\infty) \to \mathbf{R}$. From Exercise 2.9.6 and Proposition 2.9.8 we see that

$$M_s := u\left(t - (s \wedge \widetilde{T}_x),\, x + W_{s\wedge \widetilde{T}_x}\right), \qquad 0 \le s \le t-\varepsilon$$

is a Brownian martingale, where $0<\varepsilon<t/2$ and

$$\widetilde{T}_x := \inf\{\, s\ge 0 \,|\, x + W_s = 0\,\} \tag{4.21}$$

is the first time the Brownian Motion process $\mathcal{W}$ hits the site $-x$. (We employ the usual convention $\inf\emptyset = +\infty$.) But the defining property (2.9.19) of a Brownian martingale gives $\mathbf{E}(M_{t-\varepsilon}) = M_0$, or equivalently

$$\begin{aligned} u(t,x) &= \mathbf{E}\left[u\left(t-(t-\varepsilon)\wedge\widetilde{T}_x\,,\, x + W_{(t-\varepsilon)\wedge\widetilde{T}_x}\right)\right] \\ &= \mathbf{E}\left[u\left(t-\widetilde{T}_x,\, x+W_{\widetilde{T}_x}\right)\cdot\chi_{\{\widetilde{T}_x<t-\varepsilon\}} + u(\varepsilon, x+W_{t-\varepsilon})\cdot\chi_{\{\widetilde{T}_x\ge t-\varepsilon\}}\right]. \end{aligned}$$

Then, upon letting $\varepsilon\downarrow 0$, our initial and boundary conditions $\lim_{\substack{t\downarrow 0\\ y\to x}} u(t,y)=f(x)$ and $u(s,0)=g(s)$ lead to the following representation

$$u(t,x) = \mathbf{E}\Big[g(t-\widetilde{T}_x)\cdot\chi_{\{\widetilde{T}_x<t\}} + f(x+W_t)\cdot\chi_{\{\widetilde{T}_x\ge t\}}\Big] \tag{4.22}$$

of the solution to the Initial/Boundary Value Problem with Dirichlet boundary data, in terms of standard Brownian Motion.

To wit: suppose we have a semi-infinite rod, along which we specify an initial (spatial) temperature profile $f(\cdot)$; we also specify at the origin a (temporal) temperature profile $g(\cdot)$. In order to compute the temperature $u(t,x)$ at time $t>0$ and position $x>0$ we run from (t,x), and backwards in time, a Brownian Motion that gets absorbed at the origin when it hits it. We then average over the spatial temperature profile at the terminal position for those paths that do not hit the origin by time t, and over the temporal profile for those paths that do.

For instance, take $f\equiv 0$ and $g\equiv 1$. Then the expression of (4.22) gives the representation $u(t,x)=\mathbf{E}\big[\chi_{\{\widetilde{T}_x<t\}}\big] = \mathbf{P}[\widetilde{T}_x<t]$. But we already know from Exercise 4.8 that

$$u(t,x) = \int_0^t \frac{x}{s}\,p_s(0,x)\,ds = \int_0^t \frac{x}{\sqrt{2\pi s^3}}\,e^{-x^2/2s}\,ds\,.$$

Comparing the two expressions we obtain another derivation for the distribution

$$\begin{aligned} u(t,x) = \mathbf{P}[\widetilde{T}_x<t] &= \int_0^t \frac{x}{\sqrt{2\pi s^3}}\,e^{-x^2/2s}\,ds = \sqrt{\frac{2}{\pi}}\cdot\int_{x/\sqrt{t}}^\infty e^{-\xi^2/2}\,d\xi \\ &= 2\left[1-\Phi\left(\frac{x}{\sqrt{t}}\right)\right], \qquad t>0,\ x>0 \end{aligned} \tag{4.23}$$

of the first hitting time $\widetilde{T}_x$ of (4.21); recall the 'Reflection Principle' of Proposition 2.9.13 and the derivation from it of the distribution (2.9.32). By symmetry, this is also the distribution of the first hitting time $T_x := \inf\{ s \ge 0 \,|\, W_s = x \}$. Letting $t \to \infty$ in (4.23) we deduce

$$\mathbf{P}[\widetilde{T}_x < \infty] = \mathbf{P}[T_x < \infty] = 2 \cdot \lim_{t \to \infty} \left[1 - \Phi\left(\frac{x}{\sqrt{t}}\right)\right] = 1 \qquad \text{for every} \quad x > 0, \quad (4.24)$$

as well as

$$\mathbf{E}(\widetilde{T}_x) = \mathbf{E}(T_x) = \int_0^\infty \mathbf{P}(\widetilde{T}_x > t)\, dt = \int_0^\infty \left[2\Phi\left(\frac{x}{\sqrt{t}}\right) - 1\right] dt = \infty, \qquad \text{for} \quad x > 0. \quad (4.25)$$

The typical Brownian path will visit eventually any given site on the real line; but may take an awfully long time to get there. This is the **null recurrence property** of the one-dimensional Brownian Motion process, that we encountered already in Exercise 2.9.15.

4.12 Exercise: With the notation of (4.12), (4.21) show

$$\mathbf{P}[\widetilde{T}_x \ge t] = \int_0^\infty \big[p_t(x-y) - p_t(x+y) \big]\, dy \qquad \text{for every} \quad t > 0,\ x > 0.$$

(*Hint:* Take $g \equiv 0$, $f \equiv 1$ in (4.22) and recall Exercise 4.7.)

• How about the Ordinary Differential Equation $u'' - u + f = 0$ of (4.1)? Can we view its solution (4.2) also through the lens of Brownian Motion?

It turns out that here, too, there is such a connection, which passes through the so-called *resolvent*

$$(G_\alpha g)(x) := \mathbf{E} \int_0^\infty e^{-\alpha t} g(x + W_t)\, dt = \frac{1}{\sqrt{2\alpha}} \int_{-\infty}^\infty e^{-|x-y|\sqrt{2\alpha}} g(y)\, dy, \qquad x \in \mathbf{R}$$

defined for $\alpha > 0$ and any given Borel-measurable function $g : \mathbf{R} \to \mathbf{R}$ that satisfies

$$\int_{-\infty}^\infty e^{-|y|\sqrt{2\alpha}}\, |g(x+y)|\, dy < \infty, \qquad \forall\ x \in \mathbf{R}. \quad (4.26)$$

Of critical importance here, is the Laplace transform computation

$$\int_0^\infty e^{-\alpha t} \left(\frac{e^{-\xi^2/2t}}{\sqrt{2\pi t}} \right) dt = \frac{e^{-|\xi|\sqrt{2\alpha}}}{\sqrt{2\alpha}}, \qquad \xi \in \mathbf{R}.$$

It is rather easy to check that the function $G_\alpha g$ satisfies the *resolvent Ordinary Differential Equation*

$$(G_\alpha g)''(x) - 2\alpha \cdot (G_\alpha g)(x) + 2g(x) = 0 \qquad x \in \mathbf{R}, \quad (4.27)$$

provided that g is continuous and satisfies (4.26). Then, using the Markov property of Brownian motion, it can be shown that the function

$$u(x) := \mathbf{E} \int_0^\infty e^{-\alpha t - \int_0^t k(x+W_s) ds} f(x+W_t)\, dt\,, \qquad x \in \mathbf{R} \tag{4.28}$$

is the unique solution in $C^2(\mathbf{R}) \cap \mathbf{L}^1(\mathbf{R})$ of the equation

$$\frac{1}{2}\, u''(x) - \big(\alpha + k(x)\big) \cdot u(x) + f(x) = 0 \qquad x \in \mathbf{R}\,, \tag{4.29}$$

for given $\alpha > 0$, continuous $k : \mathbf{R} \to [0, \infty)$, and $f \in C(\mathbf{R}) \cap \mathbf{L}^1(\mathbf{R})$.

4.13 Exercise: Verify the claims of (4.27)-(4.29).