

## 1 Teaching notes on GMM III.

### 1.1 Variance estimation.

Most of the material in this note builds on Anderson (1971), chapters 8 and 9. [This book is now available in the Wiley Classics series].

First recall that

$$\Omega = \lim_{J \rightarrow \infty} \sum_{j=-J}^J E[f_t f'_{t-j}] .$$

Notice, that for any L dimensional vector  $a$  we have

$$a' \Omega a = \sum_{j=-J}^J a' f_t (a' f_{t-j})' ,$$

so, since the quadratic form  $\Omega$  is characterized by the bilinear mapping  $a \rightarrow a' \Omega a$  (and similar for estimates  $\hat{\Omega}$ ), you see that the behavior of the estimators are characterized by the actions of the estimator on the univariate processes  $a' f_t$ . In the following I will therefore look at the theory for spectral estimation for univariate processes, and in this section we will ignore that  $f_t$  is a function of an estimated parameter. Under the regularity conditions that is normally used, this is of no consequence asymptotically.

Defining the  $k$ 'th autocorrelation  $\gamma(k) = E f_t f'_{t-k}$ , our goal is to estimate  $\sum_{k=-\infty}^{\infty} \gamma(k)$ . Define the estimate (based on  $T$  observations) of the  $k$ 'th autocorrelation by

$$c(k) = \frac{\sum_{t=k}^T [f_t f'_{t-k}]}{T} ; k = 0, 1, 2, \dots .$$

Notice that we do not use the unbiased covariance estimate of the autocovariances (this is obtained by dividing by  $T - k$  rather than  $T$ ).

We will use estimators of the form

$$\hat{\Omega} = \sum_{j=-J}^J w_j c(k) ,$$

where the  $w_j$  are a set of weights. (The reason for these and how to choose them is the subject of most of the following). The dependence of  $f_t$  on the estimated parameter will be suppressed in the following, but it is always evaluated at our estimate.

**The spectral density is**

$$f(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma(k) \cos(\lambda k) .$$

NOTE:  $f$  now denotes the spectral density as is common in the literature, it is NOT the moment condition!!!

Now it is easy to show that

$$\int_{-\pi}^{\pi} \cos(\lambda h) f(\lambda) d\lambda = \frac{1}{2} \gamma(h) ,$$

since  $\int_{-\pi}^{\pi} \cos(\lambda h) \cos(\lambda j) d\lambda = \pi \delta_{hj}$  (where  $\delta_{hj}$  is Kronecker's delta [1 for  $h = j$ , 0 otherwise]). You can easily see that the spectral density is flat (i.e. constant) if there is no autocorrelation at all, and that  $f(\lambda)$  becomes very steep near 0, if all the autocovariances are large and positive (the latter is called the "typical spectral shape" for economic time series by Granger and Newbold). In any event, since we want to estimate only  $f(0)$ , this is the all the intuition you need about this.

### The Sample Spectral Density

Define

$$I(\lambda) = \frac{1}{2\pi} \sum_{k=-T}^T c(k) \cos(\lambda k) .$$

$I(\lambda)$  is that sample equivalent of the spectral density and is denoted the *sample spectral density*. It is fairly simple to show (you should do this !) that

$$I(\lambda) = \frac{1}{2\pi T} \left| \sum_{t=1}^T f_t e^{i\lambda t} \right|^2 .$$

The importance of this is that it shows that the sample spectral density is positive. We do not want spectral estimators that can be negative (or not positively semi-definite in the multivariate case).

Anderson (1971), p. 454 shows that

$$EI(0) = \int_{-\pi}^{\pi} k_T(\nu) f(\nu) d\nu ,$$

where

$$k_T(\nu) = \frac{\sin^2 \frac{1}{2}\nu T}{2\pi T \sin^2 \frac{1}{2}\nu}$$

is called Fejer's kernel. Notice that the expected value is a weighted average of the values of  $f(\lambda)$  in a neighborhood of 0. If the true spectral density is flat then the sample spectrum is unbiased but otherwise not in general. Anderson also shows (page 457) that if the process is normal then

$$\text{Var}(I(0)) = 2[E\{I(0)\}]^2$$

(for non-normal processes there will be a further contribution involving the 4th order cumulants).

If  $\sum |\gamma(k)| < \infty$  then one can show that

$$\lim_{T \rightarrow \infty} EI(\lambda) = f(\lambda) ,$$

and for normal processes one can show that

$$\lim_{T \rightarrow \infty} \text{Var}I(0) = 2f(0)^2 ,$$

(and again there is a further contribution from 4th order cumulants for non-normal processes).

One can also show that (for normal processes)

$$\lim_{T \rightarrow \infty} \text{Cov}\{I(\lambda)I(\nu)\} = 0 ,$$

for  $\lambda \neq \nu$ , so that the estimates for even neighboring  $\lambda$ s are independent. This independence together with the asymptotic unbiasedness is the reason that one can obtain consistent estimates of the spectral density by "smoothing" the sample spectrum.

For a general (and extremely readable) introduction to smoothing and other aspects of density estimation (these methods are not specific for spectral densities), see B. Silverman: "Density Estimation for Statistics and Data Analysis", Chapman and Hall, 1986.

### Consistent estimation of the spectral density

One can obtain consistent estimates of the spectral density function by using weights, i.e. for a sequence of weights  $w_j$

$$\hat{f}(\gamma) = \frac{1}{\pi} \sum_{r=-T+1}^{T-1} \cos(\gamma r) w_r c(r) .$$

If you define

$$w^*(\lambda|\nu) = \frac{1}{\pi} \sum_{r=-T+1}^{T-1} \cos(\lambda r) \cos(\nu r) w_r ,$$

it is easy to see that

$$\hat{f}(\nu) = \int_{-\pi}^{\pi} w^*(\lambda|\nu) I(\lambda) d\lambda .$$

We will only use these formula's for  $\nu = 0$ , but the important thing to see is that our estimate of the spectral density is a *smoothed* estimate of the sample spectral density. Also note that the usual way to show that a set of weights result in a positive density estimate is to check that the implied  $w^*(\cdot|0)$  function is positive.

Anderson (page 521) shows that

$$\lim E \hat{f}(0) = \int_{-\pi}^{\pi} w^*(\lambda|0) f(\lambda) d\lambda .$$

This means that the kernel smoothed estimate is not in general consistent for a fixed set of weights. Of course if the true spectral density is constant the smoothed estimate will be consistent (since the weights will integrate to 1 in all weighting schemes you would actually use), but the more “steep” the actual spectral density is, the more bias you would get. We will show how one can obtain an asymptotically unbiased estimate of the spectral density by letting the weights be a function of  $T$ , but the above kind of bias is still what you would expect to find in finite samples, which is why it is worth keeping in mind.

In most cases the weights take the form

$$w_j = k\left(\frac{j}{K_T}\right) ,$$

where  $k$  is a continuous function,  $k(0) = 1$ ,  $k(x) = k(-x)$ , normalized such that the implied  $w^*$  satisfies  $\int_{-\pi}^{\pi} w^*(\lambda|\nu) d\lambda = 1$  for all  $\nu$ . We will always assume that  $K_T$  tends to infinity with  $T$ .

For the asymptotic theory the smoothness of the function  $k$  near 0 is important, define  $k_q$  as

$$\lim_{x \rightarrow 0} \frac{1 - k(x)}{|x|^q} = k_q ,$$

where  $q$  is the largest exponent for which  $k_q$  is finite. Various ways of choosing the function  $k$  to generate the weights result in different values of  $q$  and  $k_q$ . Under regularity conditions (most importantly  $\sum_{r=-\infty}^{\infty} |r|^q \gamma(k) < \infty$ ) you find that for  $K_T \rightarrow \infty$  such that the  $q$ -th power grows slower than  $T$ ,  $K_T^q/T \rightarrow 0$ , then

$$\lim K_T^q [E\hat{f}(\nu) - f(\nu)] = \frac{-k_q}{2\pi} \sum_{r=-\infty}^{\infty} |r|^q \cos(\nu r) \gamma(k) .$$

Note that this implies that the smoothed estimate is consistent, and the most important is the *rate* of convergence, which is faster the larger  $K_T^q$  (subject to being less than  $T$ ).

It is easy to verify that  $q = 1$  for the Bartlett kernel, and  $q = 2$  for most other kernel schemes used. For the variance one can show that

$$\lim_{T \rightarrow \infty} \frac{T}{K_T} \text{var}\{\hat{f}_T(0)\} = 2f^2(0) \int_{-1}^1 k^2(x) dx$$

(for the estimate at points not equal to zero or  $\pi$  the factor 2 disappears - this is due to the fact that the spectral density is symmetric around 0, so at 0 a symmetric kernel will in essence smooth over only half as many observations of the sample spectral density). So we notice that the variance does not go to zero at the usual parametric rate  $\frac{1}{T}$ , but only at the slower rate  $K_T/T$ . So in order to get low variance you would like  $K_T$  to grow very slowly, but in order to obtain low bias you would like  $K_T$  to grow very fast. You can also see that asymptotically the kernel with higher values of  $q$  will totally dominate the ones with lower values of  $q$  since you for the same order of magnitude of the variance get a lower order of magnitude of the bias. In practice this may not be so relevant, however, since the parameter  $q$  only depends on the kernel near 0, which only really comes into play in extremely large samples.

Andrews (1991) shows the consistency of various kernel smoothed spectral density estimates (at 0 frequency), when the covariances are estimated via estimated orthogonality conditions (or as you would usually say, when you use the error terms rather than the unobserved innovations). In this case some more regularity conditions, securing that the error term varies smoothly with the estimated parameters, are clearly necessary.

The only kernels that allow for a  $q$  larger than 2 are kernels that do not necessarily give positive density estimates, which people tend to avoid (although Lars Hansen have used

the truncated kernel, which belongs to those). Among the kernels that have  $q = 2$  Andrews show that the optimal kernel is the one which minimizes  $k_q^2(\int_{-1}^1 k^2(x)dx)^4$ . (See Andrews (1991), Theorem 2, p. 829). This turns out to be minimized by the Quadratic Spectral (QS) kernel which has the form

$$k_{QS}(x) = \frac{25}{12\pi^2 x^2} \left( \frac{\sin(6\pi x/5)}{6\pi x/5} - \cos(6\pi x/5) \right) .$$

The usual way the bias and the variance is traded off is by minimizing the asymptotic Mean Square Error. For simplicity define

$$f^{(q)} = \frac{1}{2\pi} \sum_{r=-\infty}^{\infty} |r|^q \gamma(r) .$$

It is simple to show that the MSE is

$$\frac{K_T}{T} f^2(0) \int_{-1}^1 k^2(x)dx + \left( \frac{1}{K_T^q} \right)^2 k_q^2 [f^{(q)}]^2$$

Now in order to minimize the MSE, differentiate with respect to  $K_T$ , set the resulting expression equal to 0, solve for  $K_T$  and obtain

$$K_T = \left( \frac{2q k_q^2 [f^{(q)}]^2}{f(0)^2 \int k^2} \right)^{\frac{1}{2q+1}} T^{\frac{1}{2q+1}}$$

For example for the Bartlett kernel you can find  $k(0) = 1$  and  $\int k^2 = 2/3$ . Andrews define

$$\alpha(q) = \frac{2[f^{(q)}]^2}{f(0)^2}$$

and the optimal bandwidth

$$K_T^* = \left( \frac{q k_q^2}{\int k^2(x)dx} \right)^{\frac{1}{2q+1}} (\alpha(q)T)^{\frac{1}{2q+1}}$$

so you find

$$K_T^* = 1.1447[\alpha(1)T]^{\frac{1}{3}}$$

for the Bartlett kernel, and

$$K_T^* = 1.3221[\alpha(2)T]^{\frac{1}{5}}$$

for the QS kernel. The  $\alpha$  parameter depends on the (unknown) spectral density function at frequency 0, but Andrews suggest that one assume a simple form of the model, e.g. an AR(1) or an ARMA(1,1), or maybe a VAR(1) in the vector case, and use this to obtain an initial estimate of  $f(0)$  which one then uses for an estimate of the  $\alpha$  parameter. Notice that the important thing here is to get the order of magnitude right, so it is not necessary that the approximating AR(1) (say) model is the “correct” model. In case you knew the correct parametric model for the long run variance you would obtain more efficiency using this model directly rather than relying on non-parametric density estimators. In any event you can show for example for an AR(1) model with autoregressive parameter  $\rho$  that

$$\alpha(1) = \frac{4\rho^2}{(1-\rho)^6(1+\rho)^2} / \frac{1}{(1-\rho)^4} .$$

You should derive the one given here in order to get a feel for it (we will do it in class if there is time). More formulas are given in Andrews (1991), you will need for example  $\alpha(2)$  to use the QS kernel. Andrews also gives formulas for both  $\alpha(1)$  and  $\alpha(2)$ , for the case where the approximating model is chosen to be an ARMA(1,1), an MA of arbitrary order or a VAR(1) model. Typically the simple AR(1) model is used.

In a typical GMM application you would run an initial estimation, maybe using the identity weighting matrix, then you would obtain an estimate of the orthogonality conditions (in other word, you would get some error terms) and on those you would estimate an AR(1) model, obtaining an estimate  $\hat{\rho}$ , and you would then find

$$\hat{\alpha}(1) = \frac{4\hat{\rho}^2}{(1-\hat{\rho})^6(1+\hat{\rho})^2} / \frac{1}{(1-\hat{\rho})^4} .$$

which you would plug into your formula for the optimal bandwidth [this would be for the Bartlett kernel, for the QS kernel you would obviously have to find  $\alpha(2)$ ].

Usually you will have multivariate models and you would have to estimate either a multivariate model for the noise (e.g. a VAR(1)), although I personally estimate an AR(1) for each component series and then use the average (i.e. setting the weights  $w_a$  in Andrews’ article to 1) - this is the way the GMM program that I gave you is set up.

In my experience, the choice between (standard) k-functions matters little, while the choice of band-width ( $K_T$ ) is important. I am not quite sure how much help the Andrews’ formulae are in practice, but at least they have the big advantage that if you use a formula then the reader know you didn’t data mine  $K_T$ .

### Pre-whitening

Since the usual weighting scheme gives the autocorrelations less than full weight it is easy to see, in the situation where they are all positive, that the spectral density estimate is always biased downwards. Alternatively, remember that the spectral density estimate is a weighted average of the sample spectral density for neighboring frequencies, so if the sample spectral density is not “flat”, the smoothed estimate is biased. Therefore Andrews and Monahan (1992) suggest the use of so-called “pre-whitened” spectral density estimators. The idea is simple (and not new - see the references in Andrews and Monahan) - if one can perform an invertible transformation that makes the sample spectrum flatter, then one should do that, then use the usual spectral density estimator, and finally undo the initial transformation. This may sound a little abstract but the way it is usually implemented is quite simple: Assume you have a series of “error” terms  $f_t$  and you suspect (say) strong positive autocorrelation. Then you may want to fit an VAR(1) model (the generalization to higher order VAR models is trivial) to the  $f_t$  terms and obtain residuals, which we will denote  $f_t^*$ , i.e.

$$f_t = \hat{A}f_{t-1} + f_t^* .$$

More specifically the process of finding the  $f_t^*$ s from the  $f_t$  is denoted pre-whitening. It is easy to see that in large samples this implies (approximately)

$$(I - \hat{A})\frac{1}{T}\sum_1^T f_t = \frac{1}{T}\sum_1^T f_t^* ,$$

so we see that

$$\text{Var}\left\{\frac{1}{T}\sum_1^T f_t\right\} = (I - \hat{A})^{-1}\text{Var}\left\{\frac{1}{T}\sum_1^T f_t^*\right\}(I - \hat{A}')^{-1} ,$$

and to find your estimate of  $\text{Var}\left\{\frac{1}{T}\sum_1^T f_t\right\}$  you find an estimate of  $\text{Var}\left\{\frac{1}{T}\sum_1^T f_t^*\right\}$  and use this equality. This is denoted “re-coloring”. The reason that this may result in less biased estimates is that  $f_t^*$  has less autocorrelation and therefore a flatter spectrum around 0. On the other hand the pre-whitening operation may add more noise and one would usually only use pre-whitening in the situation where strong positive auto-correlation is expected. Also be aware that in this situation the VAR estimation is not always well behaved and you may risk that  $I - \hat{A}$  will be singular. Therefore Andrews suggests that one use a singular value decomposition of  $\hat{A}$  and truncate all eigenvalues larger than .97 to .97 (and less than -.97 to -.97) - see Andrews and Monahan (1992) for the details.

Andrews and Monahan supply Monte Carlo evidence that shows that for the models they

consider, pre-whitening results in a significant reduction in the bias, at the cost of an increase (sometimes a rather large increase) in the variance. In many applications you may worry more about bias than variance of your t-statistics, and pre-whitening may be preferred.

### **An alternative endogenous lag selection scheme**

In a recent paper Newey and West (1994) suggest another method of choosing the lag length endogenously. Remember that the optimal lag-length depends on

$$\alpha(q) = 2 \left( \frac{f^{(q)}}{f(0)} \right)^2 .$$

Newey and West suggest estimating  $f^{(q)}$  by

$$\hat{f}^{(q)} = \frac{1}{2\pi} \sum_{r=-n}^n |r|^q c(r)$$

which you get by taking the definition and plugging in the estimated autocorrelations and truncating at  $n$ . Similarly they suggest

$$\hat{f}(0) = \frac{1}{2\pi} \sum_{r=-n}^n c(r) .$$

Note that this is actually the truncated estimator of the spectral density that we want to estimate but they suggest only to use this estimate in order to get

$$\hat{\alpha}(q) = \left( \frac{2\hat{f}^{(q)}}{\hat{f}(0)} \right)^2 ,$$

and then proceed to find the actual spectral density estimator using a kernel which guarantees positive semi-definiteness. Newey and West show that one has to choose  $n$  of order less than  $T^{2/9}$  for the Bartlett kernel and order less than  $T^{2/25}$  for the QS kernel. Note that there still is an arbitrary constant (namely  $n$ ) to be chosen, but one may expect that the Newey-West lag selection scheme will be superior to the Andrews scheme in very large samples, (if you let  $n$  grow with the sample) since it does not rely on an arbitrary approximating parametric model. In Newey and West (1994) they perform some Monte Carlo simulations, that show that their own lag selection procedure is superior to Andrews' but only marginally so. In the paper Andersen and Sørensen (1996) we do, however, find a stronger preference for the Newey-West lag selection scheme in a model with high autocorrelation and high kurtosis.