Time-series properties of returns / random walks

Suppose there are 2 dates, and investors' utility fn is given by:

$$\mathcal{U}(c_t, c_{t+1}) = u(c_t) + \beta E_t[u(c_{t+1})]$$

where β is the subjective discount factor that measures impatience

Now suppose there is an asset that pays a random $x_{t+1} = p_{t+1} + d_{t+1}$ next period.
The investor has endowments e_t and e_{t+1}

How much of the asset will the investor buy or sell if its price today is p_t?

$$\max_q \quad u(c_t) + E_t[\beta u(c_{t+1})]$$

s.t. $\quad c_t = e_t - p_t q$

$\quad\quad c_{t+1} = e_{t+1} + x_{t+1} q$

F.O.C.

$$-p_t u'(c_t) + E_t[\beta x_{t+1} u'(c_{t+1})] = 0$$

$$\Rightarrow \quad p_t = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} x_{t+1}\right] \qquad \underbrace{m_{t+1} = \frac{\beta u'(c_{t+1})}{u'(c_t)}}_{\text{stochastic discount factor}}$$

$$p_t = E_t[\cancel{\beta}\, m_{t+1} x_{t+1}]$$

If $p_t \neq 0$, then $E_t[m_{t+1} R_{t+1}] = 1$

Note: I derived this as the investor's optimization problem but this is true in equilibrium for the representative investor.

Let's break this expression up:

$$E_t(m_{t+1} R_{t+1}) = 1$$

Recall that

$$Cov_t(m_{t+1}, R_{t+1}) \equiv E_t(m_{t+1} R_{t+1}) - E_t(m_{t+1}) E_t(R_{t+1})$$

so...

$$E_t(R_{t+1}) = \frac{1}{E_t(m_{t+1})} - \frac{Cov_t(m_{t+1}, R_{t+1})}{E_t(m_{t+1})}$$

if there is a riskless asset that always pays off 1 unit,

$$E_t(m_{t+1} R_{f,t+1}) = 1 \implies R_f = \frac{1}{E_t m_{t+1}}$$

and we can write the expected return as

$$E_t(R_{t+1}) = R_f - R_f Cov_t(m_{t+1}, R_{t+1}) \equiv \mu_t$$

By the def'n of expectation, we must have in realizations

$$R_{t+1} = \mu_t + \varepsilon_{t+1} \qquad E_t(\varepsilon_{t+1}) = 0$$

So Excess returns must be ~~independent~~ uncorrelated over time

$$R_{t+1} - \mu_t = \varepsilon_{t+1} \qquad E_t(\varepsilon_{t+1}) = 0 \implies Cov(\varepsilon_t, \varepsilon_{t+1}) = 0$$

$$R_t - \mu_{t-1} = \varepsilon_t$$

\$64,000 question: how do we adjust for μ properly?

Look at simple cases first: constant expected returns

or Fama's weak-form efficiences

If μ_t is constant over time, then $R_t = \mu + \varepsilon_t$
and returns themselves should be uncorrelated over time.

This is the random walk hypothesis, since prices in this case are:

$$P_{t+1} = P_t + \mu P_t + \eta_{t+1}$$

Strictly speaking, prices follow a r.w. only if $\mu = 0$.
But the right way to think about it: deflated prices follow a r.w

when should we expect this to hold? short intervals
seems unlikely that μ_t would move around on a daily basis

Empirical approach: test for serial correlation at short horizons

H_0: all autocorrelations in returns are zero

Defns Given a covariance-stationary time series $\{r_t\}$
the autocovariance at lag k (k^{th} order autocov.) is

$$\gamma_k \equiv Cov(r_t, r_{t+k})$$

the autocorrelation at lag k (k^{th}-order autocorrelation) is

$$\rho_k \equiv \frac{Cov(r_t, r_{t+k})}{\sqrt{var(r_t)\, var(r_{t+k})}} = \frac{\gamma_k}{var(r_t)} = \frac{\gamma_k}{\gamma_0}$$

These can be estimated using sample moments:

$$\hat{\gamma}_k = \frac{1}{T} \sum_{t=1}^{T-k} (r_t - \bar{r})(r_{t+k} - \bar{r}) \qquad \left(\text{or sometimes } \frac{1}{T-k}\right)$$

$$\hat{\rho}_k = \hat{\gamma}_k / \hat{\gamma}_0$$

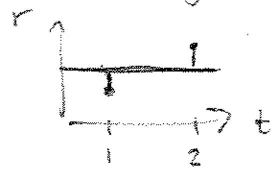
How these sample moments behave depends on true DGP.

(*) Suppose H_0 is true, and $R_t = \mu + \varepsilon_t$, $\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$ stronger than before

Then $\sqrt{T}\,\hat{\rho}_k \overset{a}{\sim} N(0,1)$, $\sqrt{T}[\hat{\rho}_1 \cdots \hat{\rho}_k]' \overset{a}{\sim} N(0, I)$
(they're independent)

In finite samples, $\hat{\rho}_k$ is always negatively biased under H_0

To see why, suppose we're just drawing a sample of $T=2$ w/ $\mu=0$

Possible outcome 

No matter what the sampling outcome, $\bar{r}$ will always be in the middle
so $Cov(r_1, r_2) < 0$ by construction

More generally, it can be shown that

$$E\hat{\rho}_k = -\frac{T-k}{T(T-1)} + O(T^{-2}) \qquad Var(\hat{\rho}_k) = \frac{T-k}{T^2} + O(T^{-2})$$

The 1st-order autocorrelation is most important:

$$E\hat{\rho}_1 = -\frac{1}{T} + O(T^{-2}) \qquad Var(\hat{\rho}_1) = \frac{T-1}{T^2} + O(T^{-2})$$

Exercise: show that $E\hat{\gamma}_1 < 0$ for finite T. (assume iid (*) above)

If $H_0 \Rightarrow$ all autocorrelations are zero, then it makes sense to aggregate the autocorrelations somehow. Box-Pierce Q-stat

$$Q_m \equiv T \sum_{k=1}^{m} \hat{\rho}_k^2 \overset{a}{\sim} \chi^2_m$$

The empirical evidence on autocorrelations:

Lots of papers, but Lo and MacKinlay (1988) is an excellent example:

	$\hat{\rho}_1$	$\hat{\rho}_2$	$\hat{Q}_5$	T
CRSP VW index '62-'94				
Daily	0.176	-0.007	263.3	8179
Weekly	0.015	-0.025	8.8	1695
Monthly	0.043	-0.053	6.8	390

Recall that, based on the definition of $\hat{\rho}_1$, it is the OLS slope in:

$$r_{t+1} = \mu + \rho_1 r_t + \varepsilon_{t+1}$$

So if the mkt goes up by 1% today, $E(r_{t+1}) = 0.176\%$ (!)

	$\hat{\rho}_1$	T	
Weekly '62-'94 Portfolios			
Smallest CRSP quintile	0.35	1695	
Middle "	0.20	1695	
Largest "	0.06	1695	
Avg. of 411 individual stocks	-0.04	1695	(hmm...)

So what must be going on? Cross-autocorrelations.

Suppose $\underset{N\times 1}{R_t}$ cov-stationary vector of returns with mean $\underset{N\times 1}{\mu}$

and autocov matrix $\underset{N\times N}{\Gamma_k} = E\left[(R_t-\mu)(R_{t+k}-\mu)'\right]$

For simplicity, consider the equal-wtd mkt return $R_{mt} = \frac{1}{N}\sum_{i=1}^{N} R_{it}$

Then

$$\mathrm{Cov}(R_{mt}, R_{m,t-1}) = \mathrm{Cov}\left(\frac{1}{N}\sum R_{it}, \frac{1}{N}\sum R_{i,t-1}\right)$$

$$= \frac{1}{N^2}\left[\underbrace{\sum_{i=1}^{N} \mathrm{Cov}(R_{it}, R_{i,t-1})}_{\substack{\text{own autocovariance} \\ N \text{ terms}}} + \underbrace{\sum_{i\neq j}\sum \mathrm{Cov}(R_{it}, R_{j,t-1})}_{\substack{\text{cross-autocovariance} \\ N^2-N = N(N-1) \text{ terms}}}\right]$$

Signing this:

$+$ $\qquad$ $-$, so $\qquad$ $+$

Weekly returns 1962-1994: $S \equiv$ smallest quintile of NYSE/AMEX stocks
$B \equiv$ biggest quintile "

Autocorrelations at lag k:

		R_{St}	R_{Bt}
$k=0$	R_{St}	1.00	0.73
	R_{Bt}	0.73	1.00
		$R_{S,t}$	R_{Bt}
$k=1$	$R_{S,t-1}$	0.352	0.024
	$R_{B,t-1}$	0.265	0.057

lead-lag pattern: large stocks lead small stocks
small stocks don't predict large stocks

Per se, this is further evidence against random walk.
But it was designed by Lo and MacKinlay (1990) to address profits in contrarian strategies

Contrarian strategies
Buy past losers
Sell past winners
often Put bigger weights on bigger extremes

See:
Lehmann (1990) – wkly
DeBondt and Thaler (1985) – 3 yr. horizon

Lehmann results: $w_{it} = R_{i,t-k} - \bar{R}_{t-k}$

weights based on Wed-Mon returns, performance measured Wed to Wed

holding period (in wks)	mean	t-stat
1	0.0121	30
4	0.0484	29
13	0.1573	27
52	0.6281	18

break-even one-way xaction costs = 0.20%

Lo and MacKinlay provide a decomposition of these profits.

Assume $w_{it} = -\frac{1}{N}(R_{i,t-1} - R_{m,t-1})$ is the pfl. wt. on asset i

and $R_{m,t-1} = \frac{1}{N}\sum R_{i,t-1}$

This is a zero-investment pfl, since the wts. sum to zero by const.

Profits $\pi_t = \sum_{i=1}^{N} w_{it} R_{it}$

Sub. in the expression for the weights and take expectations:

$$E\pi_t = -\frac{1}{N} E\left[\sum_i (R_{i,t-1} - R_{m,t-1}) R_{i,t}\right]$$

$$= -\frac{1}{N}\sum_i E(R_{i,t-1}R_{i,t}) + E(R_{m,t-1}R_{m,t})$$

$$= -\frac{1}{N}\sum_i \left[\mathrm{Cov}(R_{i,t-1}, R_{it}) + \mu_i^2\right] + \mathrm{Cov}(R_{m,t-1}, R_{m,t}) + \mu_m^2$$

$$= \underbrace{-\frac{1}{N}\sum_i \mathrm{Cov}(R_{i,t-1}, R_{it})}_{\text{own autocovariances}} + \underbrace{\mathrm{Cov}(R_{m,t-1}, R_{mt})}_{\text{mkt. autocov.}} - \underbrace{\frac{1}{N}\sum_i (\mu_i - \mu_m)^2}_{\text{cross-sectional mean dispersion}}$$

and from a couple of pages back, recall that

$$Cov(R_{mt}, R_{m,t-1}) = \frac{1}{N^2}\left[\sum Cov(R_{it}, R_{i,t-1}) + \sum_{i \neq j}\sum Cov(R_{it}, R_{j,t-1}) \right]$$

so we can write expected profits as

$$E(\pi_t) = \underbrace{C}_{\substack{\text{cross}\\ \text{auto}}} + \underbrace{O}_{\substack{\text{own}\\ \text{auto}}} - \underbrace{\sigma^2(\mu)}_{\substack{\text{dispersion}\\ \text{in means}}}$$

Weekly returns '62 - '87 (551 CRSP NYSE/AMEX firms)

	$E(\pi_t)$	%C	%O	% σ^2_μ	Avg. long position
All stocks	1.694	49.6	50.9	−0.5	152
Smallest Quintile	4.532	45.2	55.0	−0.2	209
Largest Quintile	0.617	30.5	70.3	−0.8	117

Note:

weekly profit of about 1% on long position = short position before transactions costs

Lo and MacKinlay conclude that overreaction is only part of the source of contrarian profits

- cross autocorrelations are a bigger part of it

inefficient mkts supporters: "buy it because it incorporates mkt news with a lag"

Before we start to think about why we've rejected the random walk and what it means, let's circle back and pick up one more tool for testing the null of a random walk

Variance Ratios

Under RW, variance of price increments linear in time interval
Use cts. compounded returns, and define $r_t(2) \equiv r_t + r_{t-1}$

Then the variance ratio is defined as

$$VR(2) = \frac{Var(r_t(2))}{2\,Var(r_t)} = \frac{Var(r_t + r_{t-1})}{2\,Var(r_t)}$$

$$= \frac{Var(r_t) + 2\,Cov(r_t, r_{t-1}) + Var(r_{t-1})}{2\,Var(r_t)}$$

$$= 1 + \hat{\rho}_1$$

Intuition: if prices mean revert ($\rho_1 < 0$), then the variance across two periods is smaller than the variance each period

In general:

$$VR(q) \equiv \frac{Var[r_t(q)]}{q\,Var(r_t)} = 1 + 2\sum_{k=1}^{q-1}\left(1 - \frac{k}{q}\right)\hat{\rho}_k$$

just a linear combo of autocorrs

Exer. Derive this result.

Under the null, $VR(q)$ is 1 in large samples. Sampling dist.?

H_0: $r_t = \mu + \varepsilon_t$, $\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$ $\quad \{p_0, \ldots, p_{2n}\}$

$$\hat{\sigma}_a^2 = \underbrace{\frac{1}{2n}\sum_{k=1}^{2n}(p_k - p_{k-1} - \hat{\mu})^2}_{\text{var of 1-day returns}}$$

$$\hat{\sigma}_B^2 = \underbrace{\frac{1}{2n}\sum_{k=1}^{n}(p_{2k} - p_{2k-2} - 2\hat{\mu})^2}_{\frac{1}{2} \cdot \text{var of 2-day returns}}$$

Standard results under normality:

$$\sqrt{2n}\,(\hat\sigma_a^2 - \sigma^2) \overset{a}{\sim} N(0, 2\sigma^4)$$
$$\sqrt{2n}\,(\hat\sigma_b^2 - \sigma^2) \overset{a}{\sim} N(0, 4\sigma^4)$$

$\hat\sigma_b^2$ is half as efficient as $\hat\sigma_a^2$ (makes sense — uses $\frac{1}{2}$ the data)

Hausman result:

let $\hat\theta_e$ be an asymptotically efficient estimator of θ
$\hat\theta_a$ a consistent, inefficient " "

Then, asymptotically
$$Var(\hat\theta_a - \hat\theta_e) = Var(\hat\theta_a) - Var(\hat\theta_e)$$

Outline of pf

all asymptotic

Suppose $Cov(\hat\theta_e, \hat\theta_a - \hat\theta_e) \neq 0$. Then $\exists$ l.c. of $\hat\theta_e$ and $\hat\theta_a - \hat\theta_e$ s.t. $var(\alpha\hat\theta_e + (1-\alpha)(\hat\theta_a - \hat\theta_e)) < var(\hat\theta_e)$. But this contradicts efficiency of $\hat\theta_e$. So they're uncorrelated, which means that

$$Var(\hat\theta_a) = Var(\hat\theta_e + \hat\theta_a - \hat\theta_e)$$
$$= Var(\hat\theta_e) + Var(\hat\theta_a - \hat\theta_e) \quad \text{(no cov's)}$$
$$\Rightarrow Var(\hat\theta_a - \hat\theta_e) = Var(\hat\theta_a) - Var(\hat\theta_e)$$

How do we use this here? $\hat\sigma_a^2$ is asymptotically efficient, so

variance difference
$$VD(2) \equiv Var(\hat\sigma_b^2 - \hat\sigma_a^2)$$
$$\sqrt{2n}\, VD(2) \overset{a}{\sim} N(0, 2\sigma^4)$$

\# obs

and

variance ratio
$$VR(2) = \frac{VD(2)}{\hat\sigma_a^2} + 1, \qquad \sqrt{2n}\, VR(2) \overset{a}{\sim} N(1, 2)$$

here, these are equivalent. VR theoretically more appealing
VD more robust

See CLM | Easy to generalize to $VR(q)$
| Easy to include overlapping q-period returns for ↑ efficiency

If volatility changes predictably over time, our null to date is too harsh

Even under weaker null of uncorr'd returns, $VR(q) \to 1$

One approach: model heteroskedasticity using GARCH or something else

Alternatively: use heteroskedasticity-consistent methods of White (1980)

See CLM p. 54 - 55.

Empirical results

	Weekly '62 - '94 pfls	$q=2$	$q=4$	$q=8$	$q=16$
none sig	VW return	1.02	1.02	1.04	1.02
all sig.	smallest quintile (EW)	1.35*	1.77*	2.24*	2.46*
none sig	largest quintile (EW)	1.06~~*~~	1.10~~*~~	1.14~~*~~	1.11
	avg. over individual securities (EW)	0.96	0.92	0.89	0.85

Exercise: Replicate all the empirical work discussed so far today on data from 1995 - 1999 (available on website)

Why these rejections? Possible contributors are
(1) nonsynchronous trading
(2) bid-ask bounce
Neither is economically very interesting.

<u>NST</u>

Some stocks don't trade very often. if their returns are recorded as zero when they don't trade, they may appear to lag, when in reality it's the artificial censoring process

A simple model:

<u>true</u>: $R_{it} = R_{mt} + \varepsilon_{it} \qquad i=1,\dots,N$

observed:
$$R^o_{it} = \begin{cases} 0 & \text{w/ prob } p \quad (\text{non-trading}) \\ \underbrace{P_{it}/P^o_{it-1}}_{\text{stock price "catches up" to true value}} & \text{w/ prob } 1-p \quad (\text{trading}) \end{cases}$$

Bernoulli trials are iid across firms and over time.

So, for large N the observed equal-wtd mkt return is:

$$R^o_{mt} \doteq (1-p)R_{mt} + p(1-p)R_{m,t-1} + p^2(1-p)R_{m,t-2} + \dots$$

<u>Intuition</u>:

Among other things, this implies that:

$p_j = ~~\ldots~~ \; p^j$

Induced autocorrelations in pfl returns

p	ρ_1	ρ_{weekly}
.10	0.10	0.02
.30	0.30	0.08
.50	0.50	0.17

Exer: Suppose one pfl has nontrading prob p_i and another has prob p_j

Fill in: the autocorr matrix

	$R_{i,t}$	$R_{j,t}$
$R_{i,t-1}$	p_i	?
$R_{j,t-1}$	?	p_j

So... even at unrealistic nontrading prob's, the autocorrs are too low
But this goes in the right direction.
Is there any way to tease more out of NST?

BRW (1994)
Same idea, but allow some heterogeneity

true: $R_{it} = \mu_i + \beta_i \underbrace{\Lambda_t}_{\text{common factor}} + \varepsilon_{it}$

Prob(no trade) $= p_i$, same censoring as before

[Table 3]

Conclusions:

heterogeneity gets you a lot closer to actual p's,
  but still not all the way there
still may be some evidence of slow adjustment
may not be economically inefficient
  (profits < transaction costs)
or there could be some sort of risk-based explanation
  (tho' I doubt it at these short horizons)